Abstract—Starting point is the definition of networks as ordered pairs of a skeleton and a constitutive relation. The skeleton describes the topological structure of a network. The constitutive relation describes the physical properties assigned to its branch set. The behavior of a network is defined as the set of all signal pairs obeying its constitutive relation and both Kirchhoff’s laws.

Multiports are introduced as ordered pairs consisting of a network and a family of terminal classes. The terminal classes are disjoint subsets of the node set of the corresponding network. Multiports are defined as multiports whose terminal classes contain exactly two nodes.

Based on these concepts a general theory of the terminal behavior of multiports is developed.

I. INTRODUCTION

The theory of electrical networks was at first developed as a theory of mathematical models for the analysis and synthesis of electrical circuits. Beyond that this theory also supplies tools for the solution of a variety of other technical and physical problems.

The development of computer-aided methods for the design of integrated circuits has substantially contributed to a stronger formalization of network theory.

Apart from the notions network, network analysis and isomorphisms between networks, the decomposition and interconnection of networks and the theory of terminal behavior are central parts of network theory.

The terminal behavior describes the influence of a given network on arbitrary external networks, if the given network is connected at a distinguished subset of its node set, called its terminal set, with such an external network.

The theory of terminal behavior builds together with its close relatives, network synthesis and network modelling, the bridge between network theory as a pure mathematical discipline and the applications of network theory in engineering, physics, etc.

Moreover, the theory of terminal behavior characterizes the limitations of modelling capability of networks.

Many well-known theorems of network theory describe properties of the terminal behavior of networks, e.g., the theorem on Δ-Y-transformation, the source shifting theorems going back to BLAKESLEY [7], cf. [85], the theorems of THÉVENIN and NORTON (or to be historically correct, the theorems of HELMHOLTZ and MAYER, cf. [43], [42]), [28], [19], [83], statements about the properties of interconnections of passive or reciprocal networks, e.g. [20], [28], [41] and many others. The methods of diakoptical analysis [49], [18] of large-scaled circuits are also justified by means of the theory of terminal behavior.

An appropriate explication of the notions of multiports and multiports is a problem closely related to the notion of terminal behavior. Most textbooks give only vague explanations in this context. Difficulties already arise from simple questions such as: Is each four-pole a two-port? What is really meant by an intrinsic multipole or multiport?

We develop in the following a formalization of network theory inside of set theory, i.e., networks, multiports, etc. are defined as objects of set theory. As such objects we use, besides sets, ordered pairs and ordered triples of sets, etc. The objects of network theory are then separated from the class of all these set theoretical objects by means of some defining conditions. These conditions are the axioms of our approach to network theory.

Central definitions and theorems are illustrated by simple examples.

For different approaches to the foundations of network theory we refer to [86].

For some notations of set and graph theory we refer to the Appendices A and B, and also to [30], [39], [96], [99], [104], [78].

II. NETWORKS

As in [86] networks are defined as ordered pairs of a skeleton and a constitutive relation. The skeleton, in turn, is defined as an ordered pair of two oriented graphs with the same branch and node set differing at most with respect to their orientations. It describes the topological structure of a network. The constitutive relation is a binary relation. It consists of so called signal pairs, i.e., of ordered pairs of voltages across the branch set of the network under consideration and of currents through this branch set. The constitutive relation characterizes the physical properties assigned to the branch set of a network. It is independent of the incidence relations between its branches and nodes. The constitutive relation will be introduced as a subset of a so called universal signal set. The behavior of a network, also denoted as its solution set, is the set of all signal pairs fulfilling its constitutive relation and both Kirchhoff’s laws.
A. Universal Signal Sets

The following definition delivers the basis for our formalization of network theory.

**Definition II-A.1** $\mathcal{U}$, $\mathcal{I}$, $\mathcal{P}$, and $\mathcal{T}$ denote one-dimensional normed oriented real vector spaces. $\text{Int}\mathcal{T}$ denotes the set of all intervals of $\mathcal{T}$ and $\mathcal{B}$ denotes a nondegenerated bilinear map $\mathcal{B}: \mathcal{U} \times \mathcal{I} \rightarrow \mathcal{P}$.

$\mathcal{U}$ is referred to as the space of branch voltage values, $\mathcal{I}$ as the space of branch current values, $\mathcal{P}$ as the space of power values, and $\mathcal{T}$ the time axis. The mapping $\mathcal{B}$ is the power product. □

**Examples II-A.2** The standard interpretation, defining the class of electrical networks, is specified by the assignments $\mathcal{U} := \mathbb{RV} := \{\alpha V | \alpha \in \mathbb{R}\}$, $\mathcal{I} := \mathbb{RA}$, $\mathcal{P} := \mathbb{RW}$, $\mathcal{T} := \mathbb{Rs}$, and $\forall (U,I) \in \mathcal{U} \times \mathcal{I}$ $\mathcal{B}(U,I) := U I$, where $V$, $A$, $W$, and $s$ denote the SI-units for voltage, current, power, and time, resp. The algebraic structures of $\mathcal{U}$, $\mathcal{I}$, $\mathcal{P}$, and $\mathcal{T}$ are then defined by a restriction of the usual algebraic structure of these sets of scalar physical quantities (cf. [58] and references given there) to that of a real linear space. Accordingly, $\mathcal{U}$ denotes the usual product of the scalar physical quantities $U$ and $I$. The orientations of these spaces are defined by the agreement that the units $V$, $A$, $W$, and $s$ specify positively oriented bases of the corresponding linear spaces. Their norms are defined by $\forall \alpha \in \mathbb{R} \parallel \alpha V \parallel := |\alpha|$, etc.

The assignments $\mathcal{U} := \mathcal{I} := \mathcal{P} := \mathcal{R}$ and $\mathcal{B}(U,I) := UI$ for all $(U,I) \in \mathcal{U} \times \mathcal{I}$ lead to the class of normalized networks.

But even in the class of electrical networks it is advantageous to use in addition to the standard interpretation of these spaces some others, too. As an example we mention the computation of partial derivatives of the solutions of an $\text{RLC}$-network with respect to its parameters by means of a sensitivity network, cf. [22]. If the parameter under consideration is a resistance value, then the spaces $\mathcal{U}$, $\mathcal{I}$, and $\mathcal{T}$ of the corresponding sensitivity network are defined with $\Omega := VA^{-1}$ in a natural manner by $\mathcal{U} := \mathbb{RA} = \mathbb{RV} V^{-1}$, $\mathcal{I} := \mathbb{RV} V^{-1}A^2 = \mathbb{RA} \Omega^{-1}$, $\mathcal{T} := \mathbb{Rs}$. But, for $\mathcal{P}$ no such natural choice exists. It is possible to introduce the power product by $\forall \alpha, \beta \in \mathbb{R} \mathcal{B}(\alpha A, \beta V^{-1}A^3) := k \alpha \beta V^{-1}A^3$, where $k$ is a positive real number, and according to that $\mathcal{P} := \mathbb{RV} V^{-1}A^3$. Yet with the same right we could choose $\mathcal{P} := \mathbb{RA}^2 = \mathbb{RV} A \Omega^{-1}$ and $\forall \alpha, \beta \in \mathbb{R} \mathcal{B}(\alpha A, \beta V^{-1}A^3) := k \alpha \beta A^2$. Other examples of this kind are applications of networks for modeling mechanical, chemical or thermal systems, etc. (cf. [48], [69], [92], [113], [44]).

The setting up of state space equations of RLC networks leads also to such examples. In [29] a given network is for this purpose decomposed first into a resistive, an inductive and a capacitive subnetwork. In a next step generalized resistive networks are assigned to these inductive and the capacitive subnetworks. The spaces of branch voltage and branch current values of these generalized resistive networks are defined by $\mathcal{U} := \mathbb{RV}s$, $\mathcal{I} := \mathbb{RA}$ or $\mathcal{U} := \mathbb{RV}$, $\mathcal{I} := \mathbb{RAs}$, resp. An analysis of these networks delivers finally the state space equations of the given RLC network.

Moreover, for some of such interdisciplinary applications of network theory it is expedient to replace the one-dimensional spaces $\mathcal{U}$, $\mathcal{I}$, and $\mathcal{P}$ by families of such spaces to cover models for electro-mechanical systems, etc. Furthermore, there are applications where it is necessary to replace the spaces $\mathcal{U}$ and $\mathcal{I}$ by multi-dimensional spaces or by families of such spaces (cf. [1], [88], [90]). □

Of course, if the spaces $\mathcal{U}$, $\mathcal{I}$, $\mathcal{P}$, and $\mathcal{T}$ are not specified by their standard interpretation, then their elements are to be considered as generalized voltage, current, power or time values, resp.

The linear structures of $\mathcal{U}$ and $\mathcal{I}$ are required to define linear and nonlinear networks. The orientations of $\mathcal{U}$, $\mathcal{I}$, $\mathcal{P}$, and $\mathcal{T}$ allow to distinguish positive from negative values of their elements. The norms of these spaces are indispensable if differential or integral equations are used as constitutive equations.

**Definition II-A.3** Let $\text{Int}\mathcal{T}$ denote the set of all intervals of the time axis. If $\mathcal{Z}$ is a finite set, then the set $\mathcal{S} := \bigcup_{T \in \text{Int}\mathcal{T}} (\mathcal{U}_T^2 \times (\mathcal{I}_T^2)^T)$ is denoted as the universal signal set on $\mathcal{Z}$. □

**Agreement II-A.4** For every nonvoid finite set $\mathcal{Z}$ and each interval $T \in \text{Int}\mathcal{T}$ we assume that the sets $\mathcal{U}_T^2$, $(\mathcal{U}_T^2)^T$, $\mathcal{U}_T^2 \times \mathcal{I}_T^2$, $(\mathcal{U}_T^2)^T \times (\mathcal{I}_T^2)^T$ are endowed with the structure of normed linear spaces induced by the structures of the spaces $\mathcal{U}$ and $\mathcal{I}$. □

Obviously, it holds $\dim \mathcal{U}_T^2 = \dim (\mathcal{I}_T^2)^T = |\mathcal{Z}|$.

In the following the set $\mathcal{Z}$ is always interpreted as the branch set of a network.

**Definition II-A.5** Let $\mathcal{S}$ be a universal signal set on some nonvoid finite set $\mathcal{Z}$.

The elements of $\mathcal{S}$ are called signal pairs. Let $(u, i)$ be an element of $\mathcal{S}$. Then $u$ is denoted as the associated voltage across $\mathcal{Z}$ (or shortly as a voltage) and $i$ as the associated current through $\mathcal{Z}$ (or shortly as a current). The interval $\text{dom} u = \text{dom} i$ is called the domain of the signal pair $(u, i)$. □

The consideration of signal pairs $(u, i)$ with $\text{dom} u \neq \mathcal{T}$ is necessary to cover networks with finite-escape time solutions or impasse points, cf. [21], [41].

The denotation of the time functions $u$ and $i$ as a voltage across $\mathcal{Z}$ or a current through $\mathcal{Z}$ goes back to [59].

Signal pairs are by definition ordered pairs of time functions $u$ and $i$ with the same domain $T \in \text{Int}\mathcal{T}$ and values in $\mathcal{U}_T^2$ and $\mathcal{I}_T^2$, resp. Sometimes it is convenient to assign to such a signal pair a single time function with the same domain $T$ but with function values in $\mathcal{U}_T^2 \times \mathcal{I}_T^2$.

**Definition II-A.6** Let $(u, i)$ be a signal pair of a universal signal set. Then $(u, i) \langle t \rangle$ denotes the time function $(u, i) \langle t \rangle : \text{dom} u \rightarrow \mathcal{U}_T^2 \times \mathcal{I}_T^2$ defined for all $t \in \text{dom} u$ by $(u, i) \langle t \rangle (t) := (u(t), i(t))$. □

**Definition II-A.7** Let $\mathcal{W} \subseteq \mathcal{S}$, where $\mathcal{S}$ is the universal signal set on $\mathcal{Z}$. Let $\text{sd} \mathcal{W} := \{T | \exists (u, i) \in \mathcal{W} T = \text{dom} u\}$, and let $\forall$ denote the exclusive or.
The set $\mathcal{W}$ is called restriction compatible if the following conditions are fulfilled:

(i) $\mathcal{W} \neq \emptyset \Rightarrow \bigcup_{T' \in \text{sd } \mathcal{W}} T' \in \text{sd } \mathcal{W}$,

(ii) $|\text{sd } \mathcal{W}| = 1 \forall T, T' \in \text{Int } T \ (T \in \text{sd } \mathcal{W} \land T' \subseteq T) \Rightarrow T' \in \text{sd } \mathcal{W}$,

(iii) $\forall (u, i) \in \mathcal{W} \bigcup_{T \subseteq \text{sd } \mathcal{W}} T \subseteq \text{dom } u \Rightarrow (u[T], i[T]) \in \mathcal{W}$.

The set $\mathcal{W} \subseteq \mathcal{S}$ is called properly restriction compatible if $|\text{sd } \mathcal{W}| > 1$ otherwise $\mathcal{W}$ is referred to as trivially restriction compatible. $\square$

Proposition II-A.8 If $\mathcal{W}$ is a restriction compatible subset of a universal signal set, then there exists an element $(u, i) \in \mathcal{W}$ such that $\text{dom } u \in \bigcup_{T \subseteq \text{sd } \mathcal{W}} T$. $\square$

The interval $T_{\text{max}}$ is not necessarily equal to the whole time axis. Think, for example, of a network which includes an independent voltage source whose prescribed branch voltage is proportional to a time function defined exclusively for $t \in (-\infty, t_0)$ by the assignment $t \mapsto (t_0 - t)^{-1}$ where $t_0 \in T$. Then the interval $T_{\text{max}}$ of the constitutive relation of such a network fulfills necessarily the condition $T_{\text{max}} \subseteq (-\infty, t_0)$.

Yet, restriction compatibility does not mean that each signal pair $(u, i)$ of a restriction compatible set $\mathcal{W}$ can be continued to a signal pair $(\hat{u}, \hat{i})$ defined on a larger time interval $\hat{T} \supseteq \text{dom } u$ such that $(\hat{u}, \hat{i})$ is likewise an element of $\mathcal{W}$. Typical examples for this situation are networks whose behavior includes signal pairs with forward or backward escape times or impasse points, cf. e.g. [22], p. 442, [21], [17], [23], [41], [93].

Figure 1. Examples of restrictions of signal pairs using the assignment $(u, i) \mapsto (u, i)$.

Definition II-B.1 An ordered pair $(\mathcal{G}_v, \mathcal{G}_c)$ is a skeleton if $\mathcal{G}_v$ and $\mathcal{G}_c$ are oriented graphs with the same nonvoid branch and node set, differing at most with respect to their orientation. $\mathcal{G}_v$ is denoted as the voltage graph and $\mathcal{G}_c$ as the current graph of this skeleton.

The common branch set (node set, resp.) of its graphs is also denoted as the branch set (node set, resp.) of the skeleton $(\mathcal{G}_v, \mathcal{G}_c)$. $\square$

Now we are able to introduce the notion of a network.

Definition II-B.2 An ordered pair $(\mathcal{C}, \mathcal{V})$ is a network if $\mathcal{C}$ is a skeleton and $\mathcal{V}$ is a restriction compatible subset of the universal signal set on the branch set of $\mathcal{C}$. The set $\mathcal{V}$ is referred to as the constitutive relation of $\mathcal{N}$, it is also denoted as its voltage-current relation. $\square$

Definition II-B.3 Let $\mathcal{N}$ be a network with branch set $\mathcal{Z}$ and universal signal set $\mathcal{S}$. Then $\text{pv}_b : \mathcal{U}^2 \rightarrow \mathcal{U}$ and $\text{pc}_b : \mathcal{I}^2 \rightarrow \mathcal{I}$ denote the projections defined for $U \in \mathcal{U}^2$ or $I \in \mathcal{I}^2$ by $\text{pv}_b(U) := U(b)$ and $\text{pc}_b(I) := I(b)$, resp. $\square$

Definition II-B.4 Let $\mathcal{N}$ be a network with branch set $\mathcal{Z}$ and universal signal set $\mathcal{S}$.

The time functions $u_b$ and $i_b$ defined for $b \in \mathcal{Z}$ and $(u, i) \in \mathcal{S}$ by $u_b := \text{pv}_b u$ and $i_b := \text{pc}_b i$ are denoted as the branch voltages and branch currents corresponding to the signal pair $(u, i)$.

Let $U \in \mathcal{U}^2$, $I \in \mathcal{I}^2$ and $b \in \mathcal{Z}$, then $U_b$ and $I_b$ denote the term $\text{pv}_b(U)$ and $\text{pc}_b(I)$, resp. $\square$
Definition II-B.5 Let \( \mathcal{N} \) be a network with a skeleton \((G_c, G_v)\) and branch set \( Z \), where \( A_c \) and \( A_v \) denote the incidence maps of \( G_c \) and \( G_v \), resp. Then \( Z^{ass} := \{ b \in Z \mid A_c(b) = A_v(b) \} \) is the subset of all branches of \( Z \) with associated orientations and \( Z^{opp} := \{ b \in Z \mid A_c(b) \neq A_v(b) \} \) is the subset of all branches of \( Z \) with opposite orientations. \( \square \)

The orientations of the branches in the voltage and the current graph of the skeleton of a network represent the reference directions of its branch voltages and branch currents.

The skeleton of a network is consciously introduced as an ordered pair of two oriented graphs differing at most in their orientation. This fact allows on the one hand the use of different orientations for the branch voltages and branch currents which is sometimes thoroughly reasonable (think alone on the different reference conventions used in textbooks of two-port theory). On the other hand it allows to define inside of our formalization of network theory the change of some of these orientations, cf. Subsection II-D.

Characterizing the topological structure of networks by means of oriented graphs we follow a tradition going back to G. Kirchhoff [45] and continued by W. Cauber, B. McMillan, S. Seshu et al. For a discussion of alternative approaches we refer to [86]. Our definition of the constitutive relation of a network as a set of ordered pairs of time functions, i.e., as a binary relation in the sense of set theory, was stimulated by [51] where a multiport was characterized by means of a set of so called admissible pairs of time functions fulfilling a condition which is similarly to Property (iii) of Definition II-A.7, a property which was apparently for the first time used in [123]. The notion of admissible pairs goes back to [59].

The class of Minty networks [89], [86] is a remarkable generalization of Definition II-B.2. In a Minty network the oriented graphs of its skeleton are replaced by oriented matroids introduced by G. Minty in [62]. They differ from the networks introduced in Definition II-B.2 by the fact that to each Minty network exists always a dual one.

Definition II-B.6 Let \( \mathcal{N} = ((G_c, G_v), \mathcal{V}) \) be a network with universal signal set \( \mathcal{S} \).

A signal pair \((u, i) \in \mathcal{S}\) is called a Kirchhoff signal pair of \( \mathcal{N} \) if the following two conditions hold:

(KVL) For every oriented loop \((Z^+, Z^-)\) of \( G_v \) the voltage \( u \) satisfies
\[
\sum_{b \in Z^+} u_b - \sum_{b \in Z^-} u_b = 0.
\]

(KCL) For every oriented cut set \((Z^+, Z^-)\) of \( G_c \) the current \( i \) satisfies
\[
\sum_{b \in Z^+} i_b - \sum_{b \in Z^-} i_b = 0.
\]

The conditions (KVL) and (KCL) are called Kirchhoff’s voltage law and Kirchhoff’s current law, resp. \( \square \)

In Definition II-B.6 we have used the conventions \( \sum_{b \in \emptyset} u_b := 0 \) and \( \sum_{b \in \emptyset} i_b := 0 \).

Proposition II-B.7 The condition (KCL) can be equivalently replaced by

(KCL’) For every oriented incidence cut \((Z^+, Z^-)\) of \( G_c \) the current \( i \) satisfies
\[
\sum_{b \in Z^+} i_b - \sum_{b \in Z^-} i_b = 0. \quad \square
\]

Therefore the search for criteria ensuring the existence or uniqueness of the solutions of given networks and the development of algorithms and codes for the computation of such solutions are central tasks for network theory and its application in circuit design.
Definition II-B.10 Let \( \mathcal{N} \) be a network with universal signal set \( S \) and branch set \( Z \). Furthermore let be \( W \subseteq S, T \subset \text{Int}(T) \) and \( t \in T \).

Then the set \( W_T := \{(u, i) \in W \mid \text{dom} \ u = T \} \) is the restriction of \( W \) generated by \( T \) and \( W_t := \{(u(t), i(t)) \mid (u, i) \in W\} \) is the instantaneous value relation of \( W \) at \( t \).

\[
W_c := \{u \mid \exists_t (u, i) \in W\} \quad \text{and} \quad W_c := \{i \mid \exists_u (u, i) \in W\}
\]
are denoted as domain and range of \( W \), resp.

\[
R := U^Z \times I^Z \quad \text{is the configuration space of} \quad \mathcal{N}.
\]

The sets \( W_c := \{u \mid \exists_t (U, I) \in W\} \) and \( W_c := \{I \mid \exists_u (U, I) \in W\} \) are denoted as domain and range of a set \( W \subseteq R \), resp. \( \Box \)

The subsequent proposition and its corollary follow immediately from Definitions II-B.6 and II-B.8.

Proposition II-B.11 Let \( \mathcal{N} \) be a network with universal signal set \( S \) and configuration space \( R \). Let \( \mathcal{H} \) denotes the Kirchhoff part of \( S \).

Then \( \mathcal{H} \) has the following properties:

(i) \( \mathcal{H} \) is restriction compatible.

(ii) The instantaneous value relations of \( \mathcal{H} \) are time-invariant, \( \forall t \in T \mathcal{H}_t = \mathcal{H}_0 \).

(iii) \( \mathcal{H} \) can be reconstructed from \( \mathcal{H}_0 \), i.e. \( \mathcal{H} = \{(u, i) \in S \mid \forall t \in \text{dom} \ u (u(t), i(t)) \in \mathcal{H}_0\} \).

(iv) \( \mathcal{H}_T \) is for each \( T \subset \text{Int}(T) \) a linear subspace of \( S_T \), \( \mathcal{H}_0 \) is a linear subspace of \( R \), \( \mathcal{H}_0v \) and \( \mathcal{H}_0c \) are a linear subspaces of \( U^Z \) and \( I^Z \), resp.

(v) \( \dim \mathcal{H}_0 = \frac{1}{2} \dim R \). \( \Box \)

The proof of Properties (i),..., (iv) of Proposition II-B.11 follows immediately from the the corresponding definitions, that of (v) follows from standard results of graph theory on the number of independent loops and cut sets of a graph [104].

Because the constitutive relation of a network is restriction compatible by definition, the next corollary follows directly from Proposition II-B.11.

Corollary II-B.12 The behavior of a network is restriction compatible. \( \Box \)

Let \( \mathcal{N}^{\text{phys}} \) be a physical circuit and \( (t_1, t_2, t_3, t_4) \) be a sequence of consecutive time points. Clearly, if some of the voltages and currents of \( \mathcal{N}^{\text{phys}} \) are observed between \( t_1 \) and \( t_4 \) by means of a multi-beam oscillograph and if additionally the same quantities are observed between \( t_2 \) and \( t_3 \) by means of a similar equipment, then apart from measurement deviations both observations coincide between \( t_2 \) and \( t_3 \). This fact motivates together with Corollary II-B.12 that the constitutive relation of a network has, in the general case, to meet the premiss of proper restriction compatibility.

Universal signal sets include a lot of very complicated signals, e.g. signals with nowhere differentiable voltages and currents, etc. Such signal pairs are also usually not used as models for real physical signals. The next proposition shows that one has not to worry about certain pathological signal pairs contained in \( S \), whenever the elements of the constitutive relation \( V \) are well behaved.

\[
V = \{(u, i) \in S \mid f(u, i) = g(u, i)\}
\]
holds, then the equation \( f(u, i) = g(u, i) \) is denoted as a constitutive equation of \( \mathcal{N} \). The right hand side of Equation (1) is denoted as a representation of the constitutive relation of \( \mathcal{N} \) by means of a constitutive equation. \( \Box \)

Example II-C.2 Let \( \mathcal{N} \) be a network with constitutive relation \( V \). If \( V \) or its inverse relation \( V^{-1} \) is itself a mapping, then \( i = V(u) \) and \( u = V^{-1}(i) \) are simple, but technical relevant examples of constitutive equations.

In the first case it would be possible to choose for \( E \) and \( S' \) the sets \( E := E_c \) and \( S' := \{(u, i) \in S \mid u \in E_c \land i \in S_c\} \). The functions \( f \) and \( g \) are then defined for \( (u, i) \in S' \) by \( f(u, i) := i \) and \( g(u, i) := V(u) \), resp. In the second case we can choose \( E := E_c \) and \( S' := \{(u, i) \in S \mid u \in E_c \land i \in E_c\} \), where \( f \) and \( g \) are defined for \( (u, i) \in S' \) by \( f(u, i) := u \) and \( g(u, i) := V^{-1}(u) \), resp. \( \Box \)

Already G. Kirchhoff [45] has described a network by an oriented graph and a system of constitutive equations.
Occasionally it may be useful to supplement a system of constitutive equations by means of constitutive inequalities. An interesting application of such an approach can you find in [71], [72] where an algorithm for the analysis of piecewise linear resistive networks is described which is essentially based on the use of constitutive inequalities to represent the constitutive relations of such networks.

**Proposition II-C.3** Let $\mathcal{N} := (\mathcal{C}, \mathcal{V})$ be an arbitrary network with universal signal set $S$.

Setting $S' := S \setminus \mathcal{V}$, $E := \{0, 1\}$, $g(u, i) := 1$ for all $(u, i) \in S$, $f(u, i) := 1$ for $(u, i) \in \mathcal{V}$, and $f(u, i) := 0$ for $(u, i) \in S \setminus \mathcal{V}$, then the constitutive relation of $\mathcal{N}$ can be represented by means of the constitutive equation $f(u, i) = g(u, i)$. □

Proposition II-C.3 shows that indeed any constitutive relation can be represented by means of a constitutive equation. However, the class of constitutive equations used there is only of theoretical interest since such equations are not tractable by means of standard methods of analysis and numerical mathematics. Needless to say, the representation of a constitutive relation as the solution set of a constitutive equation is by no means unique. Indeed, there is always an infinite set of such representations. Because of this fact we have introduced the notion of a network in Definition II-B.2 as an ordered pair of a skeleton and a constitutive relation and not as an ordered pair of a skeleton and a constitutive equation.

**Definition II-C.4** Let $\mathcal{N} := (\mathcal{C}, \mathcal{V})$ be a network with universal signal set $S$ whose constitutive relation is represented by means of an equating set $E$ and a constitutive equation $f(u, i) = g(u, i)$ as a subset of a set $S' := \text{dom } f = \text{dom } g \subseteq S$.

This constitutive equation is denoted as a constitutive equation in Belevitch form if the following conditions are fulfilled:

(i) The equating set $E$ is a subset of $\bigcup_{T \in \text{Int } T} (\mathbb{R}^m)^T$, where $m$ is some natural number.

(ii) The function $g$ is defined for all $(u, i) \in S'$ by $g(u, i) := 0_{\text{dom } u}$, where $0_{\text{dom } u}$ denotes the zero function of $\mathbb{R}^m_{\text{dom } u}$.

The resulting constitutive equation $f(u, i) = 0_{\text{dom } u}$ is usually simply written as $f(u, i) = 0$. □

Constitutive equations in this form are for the first time systematically considered by V. BELEVITCH for linear networks in [4], cf. [5], too.

In the standard case of nondegenerated networks the number $m$ introduced in Definition II-C.4 can be chosen equal to $\dim U^Z$.

Nullators, norators and fixators [11], [101], [110] are typical examples of degenerate networks.

The equations $i - V(u) = 0$ and $u - V^{-1}(i) = 0$, which are equivalent to that considered in Example II-C.2, are (except of an normalization factor) special cases of constitutive equations in Belevitch form. Constitutive equations in chain and hybrid form deliver further examples, cf. additionally [16], too.

Sometimes it is helpful to use for the formulation of the constitutive equations of a network, apart of voltages and currents, some additional auxiliary functions. The antiderivatives of voltages and currents, which can be interpreted as fluxes and charges, resp., are typical examples of such functions. In analogy to [118], [77] we denote such auxiliary functions as latent time-functions.

**Definition II-C.5** Let $\mathcal{N} := (\mathcal{C}, \mathcal{V})$ be a network with universal signal set $S$. Let $S'$ be a subset of $S$ with $\mathcal{V} \subseteq S'$ and $n$ be a natural number. Let $A := \bigcup_{T \in \text{Int } T} (\mathbb{R}^m)^T$ and $\mathcal{D} := \{(u, i, a) \in S' \times A \mid \text{dom } a = \text{dom } u\}$. If there exists an equating set $E$ and two mappings $f, g : \mathcal{D} \to \mathcal{E}$ such that the relationship

$$\mathcal{V} = \{(u, i) \in S' \mid \exists a \in A, f(u, i, a) = g(u, i, a)\}$$

holds, then the equation $f(u, i, a) = g(u, i, a)$ is denoted as a constitutive equation of $\mathcal{N}$ including a latent time-function. The right hand side of Equation (2) is denoted as a representation of the constitutive relation of $\mathcal{N}$ by means of a constitutive equation including a latent time-function. □

For lumped networks ordinary differential equations or differential-algebraic equations are typical examples of constitutive equations. To separate a constitutive relation from some subset of their corresponding universal signal set by means of differential or differential-algebraic equations, the signal pairs of this subset must fulfill some smoothness conditions. In most cases it is sufficiently to specify such subsets by so called signal types.

**Definition II-C.6** A set $S_s \subseteq \bigcup_{T \in \text{Int } T} U^T \times I^T$ is denoted as a signal type if it fulfills the conditions:

(i) $S_s = \{(u, i) | u \in S_{sv} \land i \in S_{sc} \land \text{dom } u = \text{dom } i\}$, where $S_{sv} := \{u | \exists_i (u, i) \in S_s\}$, $S_{sc} := \{i | \exists_u (u, i) \in S_s\}$,

(ii) $\forall_{T \in \text{Int } T} (S_s \cap U^T \times I^T$ is a linear space),

(iii) $S_s = \{(R \circ i, R^{-1} \circ u) | (u, i) \in S_s\}$, where $R : \mathcal{I} \to \mathcal{U}$ is any linear bijection,

(iv) $S_s \cap U^T \times I^T \neq \emptyset$,

(v) $\forall_{i \in \mathcal{I}} \exists_{(u, i) \in S_s} \{(u(t), i(t)) | (u, i) \in S_s\} = U \times I$,  

(vi) $\forall_{(u, i) \in S_s} \exists_{T \in \text{Int } T} (T \subseteq \text{dom } u \Rightarrow (u[T, i[T] \in S_s)$,

(vii) $\forall_{T \in \text{Int } T} \{(u \circ \text{tr}_\tau, i \circ \text{tr}_\tau) | (u, i) \in S_s\} = S_s$, where $\text{tr}_\tau : T \to T$ is defined for all values of $t, \tau \in \mathbb{R}$ by $\text{tr}_\tau(t) := t - \tau$.

Let $S_s$ be a signal type and $S$ be the universal signal set over some finite set $\mathcal{Z}$. A set $S' \subseteq S$ is called the $S_s$-signal set over $\mathcal{Z}$ if $S' = \{(u, i) \in S | \forall_{h \in \mathcal{Z}} (u_h, i_h) \in S_s\}$. □

Important examples of signal types are sets of ordered pairs of continuously differentiable, piecewise continuously differentiable functions, locally Riemann-integrable functions, etc. A simple generalization of this concept is obtained, if such a signal type is replaced by a family of signal types. In this way it would be even possible to assign different signal types to different elements of $Z$, too.

The conditions (i) – (vii) of Definition II-C.6 are motivated by the idea that network theoretical relevant properties of
a constitutive relation, such as relationships between branch voltages and currents, coupling of branches, time-invariance, reciprocity, duality, etc. should be introduced by means of the constitutive equations and not by means of a subset of the corresponding universal signal set.

Observe, a constitutive relation specified as a subset of sufficiently many continuously differentiable signal pairs by means of a system of ordinary differential equations or a mixed system of differential-algebraic equation is always restriction compatible.

**Remark II-C.7** Obviously, both Kirchhoff’s laws lead for each network \( \mathcal{N} \) to a finite system of homogeneous linear algebraic equations. Therefore, if the constitutive relation of \( \mathcal{N} \) is represented by means of some system of constitutive equations, the problem of the determination of the solution set of \( \mathcal{N} \) can be reduced to the determination of the solution set of the resulting system of equations. Such a resulting system of equations is denoted as a system of **behavioral equations** of \( \mathcal{N} \).

The introduction of node voltages, node-pair voltages, loop currents, or state coordinates as additionally auxiliary time-functions into a system of behavioral equations opens a real kaleidoscope for setting up such systems, cf. [94], [92], [112].

**Remark II-C.8** Traditionally, matrix calculus is used for the formulation of constitutive and behavioral equations of networks. Let \( \mathcal{N} \) be a network with branch set \( \mathcal{Z} \) and let \( z := |\mathcal{Z}| \). If \( \mathcal{Z} = \{(1,1),..., (z,1)\} \), then the elements of \( \mathcal{U}^\mathcal{Z} \) and \( \mathcal{I}^\mathcal{Z} \) are \( z \times 1 \) column matrices. Otherwise it is possible to assign to \( (U_b)_{b\in \mathcal{Z}} \in \mathcal{U}^\mathcal{Z} \) and \( (I_b)_{b\in \mathcal{Z}} \in \mathcal{I}^\mathcal{Z} \) by means of any bijection \( \zeta : \{1, ..., z\} \rightarrow \mathcal{Z} \) the column matrices \( U := (U_{\zeta(1)}, ..., U_{\zeta(z)})^T \) and \( I := (I_{\zeta(1)}, ..., I_{\zeta(z)})^T \), resp.

In the same manner it is possible to assign to the voltage \( u \) and the current \( i \) of a signal pair \( (u,i) \) the column matrices \( u := (u_{\zeta(1)}, ..., u_{\zeta(z)})^T \) and \( i := (i_{\zeta(1)}, ..., i_{\zeta(z)})^T \), resp.

From the viewpoint of category theory [54] the spaces \( \mathcal{U}^\mathcal{Z} \) and \( \mathcal{I}^\mathcal{Z} \) are special cases of products of vector spaces. The sets of all column matrices \( \mathcal{U} \) and \( \mathcal{I} \) assigned to the elements \( \mathcal{U}^\mathcal{Z} \) and \( \mathcal{I}^\mathcal{Z} \), resp., by means of such a bijective numbering \( \zeta \) are only other (isomorphic) representations of these spaces.

In standard software tools for computer-aided network analysis the elements of the configuration space (or of the spaces of voltage and current values) of a network with branch set \( \mathcal{Z} \) are stored in one-dimensional arrays whereas the branches of these networks are denoted by means of alphabetic names. In order to refer for data output to these branch names such tools include always subroutines realizing the corresponding canonical projections of this kinds of products. □

### D. Isomorphisms

While in the preceding parts we have considered exclusively single networks, we will now consider relationships between networks. In particular we will consider such relationships also between networks of different network classes. For this purpose we need the following definition and the next agreement.

**Definition II-D.1** Let \( \mathcal{U}, \mathcal{I}, \mathcal{T} \) and \( \mathcal{P} \) be one-dimensional oriented normed real linear spaces and \( \mathcal{B} : \mathcal{U} \times \mathcal{I} \rightarrow \mathcal{P} \) a nondegenerated bilinear mapping. Then \( \mathfrak{U}_{\mathcal{U},\mathcal{I},\mathcal{T},\mathcal{P},\mathcal{B}} \) denotes the class of all networks defined with \( \mathcal{U} \) as space of branch voltage values, \( \mathcal{I} \) as space of branch current values, \( \mathcal{T} \) as time axis, \( \mathcal{P} \) as space of power values and \( \mathcal{B} \) as power product. □

**Notations II-D.1** Let \( (\mathcal{U}, \mathcal{I}, \mathcal{T}, \mathcal{P}, \mathcal{B}) \) and \( (\bar{\mathcal{U}}, \bar{\mathcal{I}}, \bar{\mathcal{T}}, \bar{\mathcal{P}}, \bar{\mathcal{B}}) \) be two, not necessarily distinct, 5-tuples of one-dimensional oriented normed real linear spaces and nondegenerated bilinear maps \( \bar{\mathcal{B}} : \mathcal{U} \times \mathcal{I} \rightarrow \mathcal{P} \) and \( \bar{\mathcal{B}} : \bar{\mathcal{U}} \times \bar{\mathcal{I}} \rightarrow \bar{\mathcal{P}} \) such that they can be used for the definition of networks.

Then \( \mathcal{N} = (\mathcal{C}, \mathcal{V}) \) and \( \bar{\mathcal{N}} = (\bar{\mathcal{C}}, \bar{\mathcal{V}}) \) denote in the following two networks contained in \( \mathfrak{U}_{\mathcal{U},\mathcal{I},\mathcal{T},\mathcal{P},\mathcal{B}} \) and \( \mathfrak{U}_{\mathcal{U},\mathcal{I},\mathcal{T},\mathcal{P},\mathcal{B}} \), resp.

\( \mathcal{G}_v \) denotes the voltage graph, \( \mathcal{G}_c \) the current graph, \( \mathcal{Z} \) the branch set, \( \mathcal{K} \) the node set, \( \mathcal{R} \) the configuration space, \( \mathcal{S} \) the universal signal set, \( \mathcal{H} \) the Kirchhoff part of this universal signal set and \( \mathcal{L} \) denote the behavior of \( \mathcal{N} \).

Analogously, \( \bar{\mathcal{G}}_v \) denotes the voltage graph, \( \bar{\mathcal{G}}_c \) the current graph, \( \bar{\mathcal{Z}} \) the branch set, \( \bar{\mathcal{K}} \) the node set, \( \bar{\mathcal{R}} \) the configuration space, \( \bar{\mathcal{S}} \) the universal signal set, \( \bar{\mathcal{H}} \) the Kirchhoff part of this universal signal set and \( \bar{\mathcal{L}} \) denote the behavior of \( \bar{\mathcal{N}} \). □

**Definition II-D.2** A quadruple \((\rho, \tau, \pi, \sigma)\) is an isomorphism between \( \mathcal{N} \) and \( \bar{\mathcal{N}} \) if the following conditions are fulfilled:

(i) \( \rho \) is a linear bijection \( \rho : \mathcal{R} \rightarrow \bar{\mathcal{R}} \),

(ii) \( \tau \) a continuous bijection \( \tau : \mathcal{T} \rightarrow \bar{\mathcal{T}} \),

(iii) \( \pi \) a linear bijection \( \pi : \mathcal{P} \rightarrow \bar{\mathcal{P}} \) with \( \forall (U, I) \in \mathcal{R} \pi(B(U, I)) = B(\rho(U), I) \),

(iv) \( \sigma \) a bijection \( \sigma : \mathcal{S} \rightarrow \bar{\mathcal{S}} \) which maps each \((u, i) \in \mathcal{S}\) to that signal \((\bar{u}, \bar{i}) \in \bar{\mathcal{S}}\) which is uniquely defined by the condition \( \rho(\bar{u})(\tau(t)) = \rho \circ (u, i) \circ \tau^{-1} \).

(v) \( \rho(\mathcal{H}_0) = \mathcal{H}_0 \) and \( \sigma(\mathcal{L}) = \bar{\mathcal{L}} \).

If such a quadruple \((\rho, \tau, \pi, \sigma)\) exists, then network \( \mathcal{N} \) is called isomorphic to network \( \bar{\mathcal{N}} \) and vice versa. Under these assumptions, the mapping \( \sigma \) is denoted as generated by \( \rho \) and \( \tau \). □

The assignment \((u, i) \mapsto (\bar{u}, \bar{i})\) was introduced more above in Definition II-A.6.

**Example II-D.3** Let \( \mathcal{N} \) and \( \bar{\mathcal{N}} \) be normalized time-invariant resistive networks with the same skeleton and associated voltage and current reference directions. The graphs of their skeleton may consist of two branches \( a \) and \( b \) each incident with the nodes \( A \) and \( B \) and both oriented from \( A \) to \( B \). Their instantaneous value relations \( V_0 \) and \( V_0 \) may be defined as the set of all \((U, I) \in \mathcal{R} \) or \((\bar{U}, \bar{I}) \in \bar{\mathcal{R}}\) satisfying the equations \( U_a = 1, I_b = \exp(U_b) - 1 \) and \( \bar{U}_a = 1, \bar{I}_b = (e - 1)\bar{U}_b \), resp.

Obviously, both networks have for each prescribed interval \( \mathcal{T} \in \bar{\mathcal{T}} \) unique solutions \((u, i)\) and \((\bar{u}, \bar{i})\) determined with \( \text{dom} \, u = \text{dom} \, \bar{u} = \mathcal{T} \) by \( u_a(t) = \bar{u}_a(t) = 1, \bar{i}_b(t) = e - 1, (t \in \text{dom} \, u) \) and \( u_a = \bar{u}_a = u_b = \bar{u}_b; i_a = -i_b = -\bar{i}_b \).

If \( \rho, \tau \) and \( \pi \) are defined as the corresponding identical maps and \( \sigma \) is the map generated by \( \rho \) and \( \tau \), then it is easy to see that these networks are isomorphic. □
Theorem II-D.4 The network $\mathcal{N}$ is isomorphic to $\tilde{\mathcal{N}}$, if there exists a linear bijection $\rho : \mathbb{R} \to \mathbb{R}$, a continuous bijection $\tau : \mathcal{T} \to \tilde{\mathcal{T}}$ and a linear bijection $\pi : \mathcal{P} \to \tilde{\mathcal{P}}$ such that $\rho(\mathcal{H}_0) = \mathcal{H}_0$ and $\sigma(V) = \tilde{V}$, where $\sigma$ is generated by $\rho$ and $\tau$.

The proof of Theorem II-D.4 is based on the fact that for an arbitrary bijection $f : X \to Y$ and arbitrary subsets $U, V \subseteq X$ the identity $f(U \cap V) = f(U) \cap f(V)$ holds, cf. [39], [30].

The properties of $\rho$, $\tau$ and $\pi$ imply the bijectivity of $\sigma$ and therefore it follows $\sigma(\mathcal{H} \cap \tilde{\mathcal{V}}) = \sigma(\mathcal{H}) \cap \tilde{\mathcal{V}} = \mathcal{H} \cap \tilde{\mathcal{V}}$, i.e. $\sigma(\mathcal{L}) = \tilde{\mathcal{L}}$. □

Examples II-D.5 Normalization of networks, cf. [95], is a simple, nevertheless important, example of an isomorphism between networks. Let us assume that $\mathcal{N} = (\mathcal{C}, \mathcal{V})$ is an electrical network and $\tilde{\mathcal{N}} = (\tilde{\mathcal{C}}, \tilde{\mathcal{V}})$ a normalized network. And let us additionally assume that $\mathcal{N}$ and $\tilde{\mathcal{N}}$ have the same skeleton, i.e. $\mathcal{C} = \tilde{\mathcal{C}}$. Then the quadruple $(\rho, \tau, \pi, \sigma)$ is an isomorphism between $\mathcal{N}$ and $\tilde{\mathcal{N}}$ if the following conditions are fulfilled: (i) there exist positive elements $U_{\text{ref}}, I_{\text{ref}}, P_{\text{ref}}$ of $\mathcal{U}, \mathcal{I}, \mathcal{P}$ and $\rho$ and $\tau$ obey for all arguments $(\mathcal{U}, \mathcal{I})$, $(\mathcal{I}, \mathcal{U})$ and $t$ the relationships $\rho(U, I) = (\tilde{U}, \tilde{I})$ and $\tau(t) = \tilde{t}$, resp., (ii) $\sigma(V) = \tilde{V}$, where $\sigma$ is generated by $\rho$ and $\tau$. (Observe, the definition of $\sigma$ implies $\sigma(\mathcal{H}) = \tilde{\mathcal{H}}$.)

Networks with isomorphic graphs deliver another special case of an isomorphism between networks. In this case we assume that $\mathcal{N}$ and $\tilde{\mathcal{N}}$ are both elements of the same class $\mathcal{N}_{\mathcal{U}, \mathcal{I}, \mathcal{P}, \mathcal{P}}$. Then there exists an ordered pair $(\zeta, \kappa)$ which is an isomorphism between $\mathcal{G}_\kappa$ and $\tilde{\mathcal{G}}_\zeta$ as well as between $\mathcal{G}_\kappa$ and $\tilde{\mathcal{G}}_\zeta$. Under this assumption the quadruple $(\rho, \tau, \pi, \sigma)$ is an isomorphism between $\mathcal{N}$ and $\tilde{\mathcal{N}}$ if (i) $\rho$ fulfills the condition $\rho(U, I) = (U \circ \zeta^{-1}, I \circ \zeta^{-1})$, (ii) $\tau$ is the identical mapping $\tau = \text{id}_\mathcal{T}$, (iii) $\sigma$ is the mapping generated by $\rho$ and $\tau$, and (iv) $\sigma(V) = \tilde{V}$. (Observe, the definition of $\sigma$ implies here likewise $\sigma(\mathcal{H}) = \tilde{\mathcal{H}}$.)

Networks with Whitney-isomorphic graphs (cf. Appendix B) deliver a further special case of an isomorphism between networks. In this case we assume that $\mathcal{N}$ and $\tilde{\mathcal{N}}$ are both elements of the same class $\mathcal{N}_{\mathcal{U}, \mathcal{I}, \mathcal{P}}$. Then there exists a bijection $\zeta$ between their branch sets which is a Whitney-isomorphism between $\mathcal{G}_\kappa$ and $\tilde{\mathcal{G}}_\zeta$ as well as between $\mathcal{G}_\kappa$ and $\tilde{\mathcal{G}}_\zeta$. Under this assumption the quadruple $(\rho, \tau, \pi, \sigma)$ is an isomorphism between $\mathcal{N}$ and $\tilde{\mathcal{N}}$ if (i) $\rho$ fulfills the condition $\rho(U, I) = (U \circ \zeta^{-1}, I \circ \zeta^{-1})$, (ii) $\tau$ is the identical mapping $\tau = \text{id}_\mathcal{T}$, (iii) $\sigma$ is the mapping generated by $\rho$ and $\tau$, and (iv) $\sigma(V) = \tilde{V}$. (Observe, the definition of the Whitney-isomorphism of graphs implies also here $\sigma(\mathcal{H}) = \tilde{\mathcal{H}}$.)

In a similar manner it can be shown that dual networks are isomorphic. To show this, let us assume that $\mathcal{N}$ and $\tilde{\mathcal{N}}$ are dual networks. Then $(\tilde{\mathcal{G}}_\kappa, \tilde{\mathcal{G}}_\zeta)$ and $(\mathcal{G}_\zeta, \mathcal{G}_\kappa)$ are pairs of dual graphs and there exists a bijection $\zeta : \tilde{\mathcal{Z}} \to \tilde{\mathcal{Z}}$, which assigns to each oriented loop of $\mathcal{G}_\kappa$ an oriented cut set of $\tilde{\mathcal{G}}_\zeta$ and to each oriented cut set of $\tilde{\mathcal{G}}_\zeta$ an oriented loop of $\mathcal{G}_\kappa$. Let $R_{\kappa} : \mathcal{T} \to 0$ be any linear bijection and $\rho$ be the mapping defined by assignment $\rho(U, I) := (R_{\kappa} \circ 0 \circ \zeta^{-1}, R_{\kappa} \circ 0 \circ \zeta^{-1})$. Let $\pi := \text{id}_\mathcal{T}$, $\tau := \text{id}_\mathcal{T}$ and $\sigma$ be the map generated by $\rho$ and $\tau$. If $\sigma(V) = \tilde{V}$, then $(\rho, \tau, \pi, \sigma)$ is an isomorphism between $\mathcal{N}$ and $\tilde{\mathcal{N}}$. (Again, the definition of $\sigma$ implies $\sigma(\mathcal{H}) = \tilde{\mathcal{H}}$.)

A change of some branch orientations delivers a further special case of an isomorphism. Let $\mathcal{A}_\kappa$ and $\mathcal{A}_\zeta$ denote the incidence maps of $\mathcal{N}$ and $\tilde{\mathcal{N}}$, and let us assume that $\mathcal{N}$ and $\tilde{\mathcal{N}}$ have additionally the same branch and node sets. Under these assumptions the networks $\mathcal{N}$ and $\tilde{\mathcal{N}}$ have the same configuration space, the same universal signal set and the same Kirchhoff part.

By means of their incidence maps we introduce the sets $\mathcal{Z}_+: = \{ b \in \mathbb{Z} | \mathcal{A}_\kappa(b) = \mathcal{A}_\zeta(b) \}$, $\mathcal{Z}_- := \{ b \in \mathbb{Z} | \mathcal{A}_\kappa(b) \neq \mathcal{A}_\zeta(b) \}$, $\mathcal{Z}_+ := \{ b \in \mathbb{Z} | \mathcal{A}_\kappa(b) = \mathcal{A}_\zeta(b) \}$, and $\mathcal{Z}_- := \{ b \in \mathbb{Z} | \mathcal{A}_\kappa(b) \neq \mathcal{A}_\zeta(b) \}$.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be that map which is uniquely defined by the property that the relationship $\rho(U, I) = (\tilde{U}, \tilde{I})$ is fulfilled for all $(U, I) \in \mathcal{R}$ if and only if the conditions $\forall b \in \mathcal{Z}_+: \mathcal{U}_b = \tilde{\mathcal{U}}_b$, $\forall b \in \mathcal{Z}_- : \mathcal{U}_b = -\tilde{\mathcal{U}}_b$, $\forall b \in \mathcal{Z}_+: \mathcal{I}_b = \tilde{\mathcal{I}}_b$, $\forall b \in \mathcal{Z}_- : \mathcal{I}_b = -\tilde{\mathcal{I}}_b$ are met. Let $\tau := \text{id}_\mathcal{T}$, $\pi := \text{id}_\mathcal{T}$ and let $\sigma : S \to S$ be the map generated by $\rho$ and $\tau$. If $\sigma(V) = \tilde{V}$, then the networks $\mathcal{N}$ and $\tilde{\mathcal{N}}$ are isomorphic. (Also here, the definition of $\sigma$ implies $\sigma(\mathcal{H}) = \tilde{\mathcal{H}}$.) □

A generalization of the notion of isomorphic graphs was at first time systematically considered for nonoriented graphs by H. WHITNEY [117], [116]. He has denoted this generalization as a 2-isomorphism. Discussions of this notion are to be find in [99] and in particular in [78]. Apparently, there is a close connection to papers of R. M. FOSTER, cf. [34]. Based on [62] it is possible to transfer this notion to oriented graphs. Fig. 4 shows an example of networks with such Whitney-isomorphic graphs.

![Figure 4](image-url)
ideal transformer of $\mathcal{N}$ has a turns ratio $1 : n$ and that of $\mathcal{N}$ has a turns ratio $1 : 2$

Identifying the spaces $\mathcal{U}^Z = \mathcal{I}^Z$ with the space of column matrices $\mathbb{R}^{4 \times 1}$, then the instantaneous value relations $V_0$ and $V_0$ may be defined as the set of all $(U, I) \in \mathcal{R}$ or $(\bar{U}, \bar{I}) \in \mathcal{R}$ fulfilling the equation

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4
\end{pmatrix}
= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
$$

and

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\bar{U}_1 \\
\bar{U}_2 \\
\bar{U}_3 \\
\bar{U}_4
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & -4
\end{pmatrix}
\begin{pmatrix}
\bar{I}_1 \\
\bar{I}_2 \\
\bar{I}_3 \\
\bar{I}_4
\end{pmatrix}
= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
$$

resp.

Let $\rho$ be defined with $D := \text{diag}(1 \ 1 \ 2 \ 2)$ by the assignment $(U, I) \rightarrow (\bar{U}, \bar{I}) := (DU, D^{-1} I)$. If $(U, I)$ fulfills the constitutive equation of $\mathcal{N}$, then $(\bar{U}, \bar{I})$ fulfills the constitutive equation of $\mathcal{N}$, i.e. $V_0 = \rho(V_0)$. Similarly, if $U$ is a solution of a system of fundamental loop equations of $\mathcal{N}$ (and therefore of $\mathcal{N}$, too), then $\bar{U}$ is likewise a solution of this system of loop equations. The same can be shown for $I$ and $\bar{I}$ and a corresponding system of fundamental cut sets of $\mathcal{N}$ and $\mathcal{N}$. That means $\mathcal{H}_0 = \rho(\mathcal{H}_0)$. If $\pi$ and $\tau$ are chosen as the corresponding identical maps, then $(\rho, \tau, \pi, \sigma)$ is an isomorphism between $\mathcal{N}$ and $\mathcal{N}$.

**Example II-D.7** Let $\mathcal{N}$ and $\mathcal{N}$ be normalized time-invariant resistive networks with associated voltage and current reference directions as defined by the circuit diagrams shown in Fig. 6. $\mathcal{Z} = \{1, 2, 3, 4, 5\}$ and $\mathcal{Z} = \{1, 2, 3, 4, 5\}$ are the branch set of $\mathcal{N}$ and $\mathcal{N}$, resp. Their instantaneous value relations $V_0$ and $V_0$ are defined as the set of all $(U, I) \in \mathcal{R}$ and $(\bar{U}, \bar{I}) \in \mathcal{R}$ fulfilling the equations

$$U_1 = 1, \ U_3 = U_2, \ I_2 = 0, \ U_4 = I_4, \ U_5 = I_5$$

and

$$\bar{U}_1 = 1, \ \bar{I}_3 = \frac{1}{2} \bar{U}_2, \ \bar{I}_2 = 0, \ \bar{U}_4 = 4 \bar{I}_4, \ \bar{U}_5 = 4 \bar{I}_5.$$

The mapping $\rho : \mathcal{R} \rightarrow \mathcal{R}$ maps each $(U, I) \in \mathcal{R}$ to that $(\bar{U}, \bar{I})$ defined by

$$\bar{U}_1 = U_1, \ \bar{U}_2 = U_2, \ \bar{U}_3 = 2I_3, \ \bar{U}_4 = 2I_4, \ \bar{U}_5 = 2I_5$$

and

$$\bar{I}_1 = I_1, \ \bar{I}_2 = I_2, \ \bar{I}_3 = \frac{1}{2} \bar{U}_3, \ \bar{I}_4 = \frac{1}{2} I_4, \ \bar{I}_5 = \frac{1}{2} U_5.$$

If $\pi$ and $\tau$ are chosen as the corresponding identical maps, then it can be shown in the same manner as in the preceding example that $\mathcal{N}$ and $\mathcal{N}$ are indeed isomorphic. The networks $\mathcal{N}$ and $\mathcal{N}$ are to some extend related to another by a “partial dualization”. □

To include examples of this kind the notion of an isomorphism between networks introduced in Definition II-D.2 is based on a mapping $\rho : \mathcal{R} \rightarrow \mathcal{R}$ between the configuration spaces of these networks and not on two separated mappings $\rho_v : \mathcal{U}^Z \rightarrow \mathcal{U}^Z$ and $\rho_c : \mathcal{I}^Z \rightarrow \mathcal{I}^Z$.

**E. Decomposition and Interconnection of Networks**

We have introduced networks in Subsection II-B as whole entities and not as interconnections of some more elementary objects. Nevertheless the decomposition of networks into subnetworks, the representation of networks as interconnections of their subnetworks, and the replacement of subnetworks in such interconnections by means of other networks with the same terminal behavior play an important role in network theory [119]. In particular, the standard modeling strategy is based on these notions. For instance, physical electronic circuits consist of interconnections of devices with two or more terminals. If the external behavior of these devices can be described with sufficient accuracy by means of voltages and currents instead of electromagnetic fields, the modeling of such a circuit can be subdivided into two steps. In a first step the device models must be designed. Obviously, multipoles with two, three or more terminals are the candidates for device models. The number of their terminals is in general equal to the number of terminals of the corresponding devices. The terminal behavior of a device model has to deliver an (approximate) description of the relationships between the terminal-pair voltages and terminal currents which are simultaneously observable at the corresponding device terminals. In a second step the device models are interconnected at their terminals in the same manner as the devices in the given physical circuit. The analysis of the interconnections of these device models is an important basis for computer-aided circuit design. It has to
provide the designer with hints for the expected characteristics of the resulting interconnection of the real physical devices.

**Definition II-E.1** Let $\mathcal{C} := (\mathcal{G}_v, \mathcal{G}_c)$ be a skeleton with branch set $Z$ and $\emptyset \subset Z' \subset Z$. The ordered pair $\mathcal{G}_z := (\mathcal{G}_v, z, \mathcal{G}_c, z)$ is the sub skeleton of $\mathcal{C}$ generated by $Z'$ if $\mathcal{G}_v, Z'$ and $\mathcal{G}_c, Z'$ are the subgraphs of $\mathcal{G}_v$, $\mathcal{G}_c$ generated by $Z'$, resp.

For the definition of a subgraph generated by a subset of its branch set we refer to Appendix B.

**Definition II-E.2** Let $\mathcal{N}$ be a network with branch set $Z$ and universal signal set $S$ and let $Z' \subset Z$ with $Z' \neq \emptyset$.

Then $\mathcal{N}_z := (\mathcal{G}_v, z, \mathcal{N}_z)$ is the sub network of $\mathcal{N}$ generated by $Z'$ if $\mathcal{G}_v, Z'$ and $\mathcal{G}_c, Z'$ are the subgraphs of $\mathcal{G}_v$, $\mathcal{G}_c$ generated by $Z'$, resp.

The time functions $u_{Z'}$ and $i_{Z'}$ defined for each $(u, i) \in S$ by $u_{Z'} := u \cdot Z'$ and $i_{Z'} := i \cdot Z'$ are denoted as partial voltages across $Z'$ and partial currents through $Z'$.

**Definition II-E.3** Let $\mathcal{N}$ be a network with branch set $Z$ and universal signal set $S$.

For each $W \subseteq S$ and for each $Z'$ with $\emptyset \subset Z' \subset Z$ the set $W_{Z'} := \{(u, i) \in W \mid (u, i) \in W\}$ is denoted as the projection of $W$ generated by $Z'$.

**Definition II-E.4** Let $S$ be a universal signal set on some branch set $Z$ and $W \subseteq S$.

Let $(\mathcal{W}_l)_{l \in L}$ be a partition of $Z$ and $(\mathcal{W}_l)_{l \in L}$ be the family of sets $\mathcal{W}_l := W_{Z_l}$ ($l \in L$).

Then the set $V := \bigotimes_{l \in L} \mathcal{W}_l := \{(u, i) \in S \mid \forall l \in L (u, i) \in \mathcal{W}_l\}$ is denoted as the signal-set product of the $\mathcal{W}_l$ ($l \in L$).

For the special case where the index set $I$ includes only two elements we introduce for the signal-set product a simplified notation.

**Definition II-E.5** If $|I| = 2$ then the signal-set product is denoted with $\{Z', Z''\} := \mathcal{W}_{Z'_l} \otimes \mathcal{W}_{Z''_l} := \bigotimes_{l \in L} \mathcal{W}_{Z_l}$.

Clearly, it follows $\mathcal{W}_{Z_l} \otimes \mathcal{W}_{Z''_l} = \mathcal{W}_{Z'_l} \otimes \mathcal{W}_{Z''_l}$ from this definition.

Let $\mathcal{N} := (\mathcal{C}, V)$ be a network with branch set $Z$. Since its constitutive relation is restriction compatible, all its projections $\mathcal{V}_{Z_l}$ ($\emptyset \subset Z' \subset Z$) are restriction compatible, too. This fact delivers the proof of the next proposition.

**Proposition II-E.6** Let $\mathcal{N} := (\mathcal{C}, V)$ be a network with branch set $Z$ and universal signal set $S$. Let $(\mathcal{Z}_l)_{l \in L}$ be a partition of $Z$. Then the ordered pairs $\mathcal{N}_{Z_l} := (\mathcal{C}_{Z_l}, \mathcal{V}_{Z_l})$ ($l \in L$) are networks and the constitutive relation of $\mathcal{N}$ satisfies the relationship $V \subseteq \bigotimes_{l \in L} \mathcal{V}_{Z_l}$.

**Definition II-E.7** Let $\mathcal{N} := (\mathcal{C}, V)$ be a network with branch set $Z$ and universal signal set $S$. Let $(\mathcal{Z}_l)_{l \in L}$ be a partition of $Z$.

The networks $\mathcal{N}_{Z_l}$ ($l \in L$) are denoted as generated by the corresponding set $Z_l$.

**Definition II-E.8** Let $\mathcal{N} := (\mathcal{C}, V)$ be a network with branch set $Z$ and let $(\mathcal{Z}_l)_{l \in L}$ be a partition of $Z$.

The networks $\mathcal{N}_{Z_l}$ ($l \in L$) are denoted as pairwise uncoupled if between $V$ and the $\mathcal{V}_{Z_l}$ ($l \in L$) the relationship $V = \bigotimes_{l \in L} \mathcal{V}_{Z_l}$ holds.

As we have seen, for each network $\mathcal{N} := (\mathcal{C}, V)$ with branch set $Z$ the ordered pairs $(\mathcal{C}_{Z_l}, \mathcal{V}_{Z_l})$ ($\emptyset \subset Z' \subset Z$) are networks. But such an ordered pair is denoted as a subnetwork of $\mathcal{N}$ only if the sets $Z'$ and $Z'' := Z \setminus Z'$ are uncoupled.

If the constitutive relation of a network is described by means of behavioral equations, then the following proposition delivers a sufficient criterion to indicate that the elements of a partition of its branch set are pairwise uncoupled.

**Proposition II-E.9** Let $\mathcal{N} := (\mathcal{C}, V)$ be a network with branch set $Z$, universal signal set $S$, and let $(\mathcal{Z}_l)_{l \in L}$ be a partition of $Z$.

If there exist a subset $S' \subseteq S$, a family $(\mathcal{E}_l)_{l \in L}$ of equating sets, and families $(\mathcal{F}_l)_{l \in L}$ and $(\mathcal{G}_l)_{l \in L}$ of mappings $f_l : S_{Z_l} \rightarrow E$ and $g_l : S_{Z_l} \rightarrow E$ such that $V = \{(u, i) \in S' \mid \forall l \in L f_l((u, i)) = g_l((u, i))\}$, then the sets $Z_l$ ($l \in L$) are pairwise uncoupled.

It is a nice exercise to show that the branches of a voltage-controlled voltage source are coupled if it has a nonvanishing transmission factor.

**Definition II-E.10** A network with two or more branches is denoted as a non-decomposable network if it does not include any subnetwork.

With other words, a network is non-decomposable if all its branches are coupled with each other.

**Definition II-E.11** A network is denoted as an elementary network, if its graphs are forests.

May be that some components of the graphs of an elementary network consist of an isolated node only, cf. Example III-B.25 in Subsection III-B and Example III-C.22 in Subsection III-C.

After the decomposition of networks we will now consider their interconnection. To simplify the representation we introduce the following notations.

**Notations II-E.1** All networks considered in the following are of the same class.

Let $(\mathcal{N}_l)_{l \in L}$ denote a given finite family of networks. Each of the networks $\mathcal{N}_l := (\mathcal{C}_l, V_l)$ ($l \in L$) may have respectively a branch set denoted by $Z_l$, a node set denoted by $K_l$ and a universal signal set denoted by $S_l$. The incidence mappings of the voltage and current graph of $\mathcal{N}_l$ are denoted by $A_l^v$ and $A_l^c$, resp. The following abbreviations $Z := \bigcup_{l \in L} Z_l$, $K := \bigcup_{l \in L} K_l$, $A_v := \bigcup_{l \in L} A_l^v$, and $A_c := \bigcup_{l \in L} A_l^c$ are additionally used.

Furthermore, let $\mathcal{N} := (\mathcal{C}, V)$ denote a network with branch set $Z$, node set $K$ and universal signal set $S$. The incidence mappings of its voltage and current graph are denoted by $A_v$ and $A_c$, resp.
Definition II-E.12 The networks $N^l$ $(l \in L)$ are denoted as interconnectable, if $\forall k,l \in L, k \neq l (Z^k \cap Z^l = \emptyset) \land K^k \cap Z^l = \emptyset \land sdV^k = sdV^l$.

The networks $N^l$ $(l \in L)$ are skeleton disjoint if $\forall k,l \in L, k \neq l (Z^k \cap Z^l = \emptyset) \land K^k \cap Z^l = \emptyset \land K^k \cap K^l = \emptyset$. □

Remark II-E.13 Of course, if the networks of $(N^l)_{l \in L}$ are skeleton disjoint and their constitutive relations fulfill the cond-

Lemma II-E.14 If the networks $N^l$ $(l \in L)$ are interconnectable, then the unions $A_v := \bigcup_{l \in L} A_v^l$ and $A_c := \bigcup_{l \in L} A_c^l$ are mappings of $\bar{Z}$ into $\bar{K} \times \bar{K}$. □

Definition II-E.15 The network $N$ is an interconnection of the networks $N^L$ $(l \in L)$ if $Z = \bar{Z}$, $V = \bigotimes_{l \in L} V_{Z^l}$ and there exists a mapping $\kappa : \bar{K} \rightarrow K$ such that $A_v = (\kappa \times \kappa) \circ A_v^l$, $A_c = (\kappa \times \kappa) \circ A_c^l$. □

The map $\kappa \times \kappa$ is defined in Appendix B.

Definition II-E.15 is based on the idea that by an interconnection of the networks $N^L$ $(l \in L)$ the elements of some subsets of their node sets are to be identified. The complete pre-image of an element $v \in K$ under $\kappa$ is by definition a subset of $\bar{K} = \bigcup_{l \in L} K^l$. Exactly the elements of each of these subsets are identified by this interconnection of the networks $N^L$ $(l \in L)$. The assumption $\forall k,l \in L, k \neq l (sdV^k = sdV^l)$ guarantees that each network $N^l$ is isomorphic to that subnetwork of $N$ which has the same branch set as $N^l$. Therefore this condition seems to be a natural assumption in the definition of the interconnectability of networks.

Due to the degrees of freedom in the choice of the node set for network $N$, an infinite set of interconnections to a given family $(N^l)_{l \in L}$ of interconnectable networks exists. But all these interconnections are pairwise isomorphic.

We consider now some standard examples to construct interconnections of a given family of interconnectable networks.

Proposition and Definition II-E.16 Let $(N^l)_{l \in L}$ be a family of interconnectable networks, $\bar{K}$ be an arbitrary finite set with $Z \cap \bar{K} = \emptyset$ and $\kappa$ a map $\kappa : \bar{K} \rightarrow \bar{K}$.

Then the network $N$ defined by $Z := \bar{Z}$, $K := \bar{K}$, $A_v := (\kappa \times \kappa) \circ A_v^l$, $A_c := (\kappa \times \kappa) \circ A_c^l$ and $V := \bigotimes_{l \in L} V^{Z_l}$ is an interconnection of the networks $N^L$ $(l \in L)$.

The network $N$ is then denoted as generated by a node map. □

Proposition and Definition II-E.17 Let $(N^l)_{l \in L}$ be a family of interconnectable networks and $\rho \subseteq \bar{K} \times \bar{K}$ be an equivalence relation.

Then the network $N$ defined by $Z := \bar{Z}$, $K := \bar{K}/\rho$, $A_v := (\kappa \times \kappa) \circ A_v^l$, $A_c := (\kappa \times \kappa) \circ A_c^l$, where $\kappa : \bar{K} \rightarrow K$ is defined by the assignment $\kappa(v) := [v]_\rho$, and $V := \bigotimes_{l \in L} V^{Z_l}$, is an interconnection of the networks $N^l$ $(l \in L)$.

The network $N$ is then denoted as generated by an equivalence relation. □

For an example of such an interconnection we refer to Fig. 17 in Subsection III-B.

Let us now assume that the networks $N^l$ $(l \in L)$ are that subnetworks of a given network $N$ which are defined by means of a partition $(Z^l)_{l \in L}$ of the branch set of network $N$, i.e. $N^l := N_{Z^l}$. If we now construct an interconnection of these networks generated by the equivalence relation $id_K$ where $K$ denotes as mentioned above the node set of $N$. Then the node set of the interconnection of these subnetworks generated by this equivalence relation $\rho$ is equal to $\{ [v] \in K \}$. Because $\{ [v] \}$, $v \in K$, the interconnection of these subnetworks is isomorphic, but not equal to the originally given network $N$.

Proposition and Definition II-E.18 Let $(N^l)_{l \in L}$ be a family of interconnectable networks, $\rho \subseteq \bar{K} \times \bar{K}$ be an equivalence relation and $\eta : \bar{K}/\rho \rightarrow K$ a choice function, i.e., a mapping with $\forall v \in K \eta([v]_\rho) \in [v]_\rho$.

Then the network $N$ defined by $Z := \bar{Z}$, $K := dom(\rho)$, $A_v := (\kappa \times \kappa) \circ A_v^l$, $A_c := (\kappa \times \kappa) \circ A_c^l$, where $\kappa : \bar{K} \rightarrow K$ is defined by the assignment $\kappa(v) := \eta([v]_\rho)$, and $V := \bigotimes_{l \in L} V^{Z_l}$ is an interconnection of the networks $N^l$ $(l \in L)$.

The network $N$ is then denoted as generated by equivalence equation and a choice function. □

Let us now consider again the interconnection of the subnetworks $N^l := N_{Z^l}$ of $N$. If we choose $\rho := id_K$ and $\eta$ defined by $\{ v \} \mapsto v$, then the interconnection of these networks generated by this ordered pair $(\rho, \eta)$ is equal to $N$.

The definition of interconnectability does intentionally not include the condition $\forall k,l \in L, k \neq l (K^k \cap K^l = \emptyset)$ since otherwise it would be impossible to represent a network as an interconnection of its subnetworks.

Obviously, the following theorem can be proved.

Theorem II-E.19 Each network can be represented as an interconnection of elementary networks. □

Elementary networks are usually denoted in the literature as network elements. But in the formalization of network theory considered in this section they are itself networks (obviously, networks with very simple skeletons) and they can be subnetworks of other, larger networks. The relation between a given network and its subnetworks is similar to the relation between a given set and its subsets and not to that between a set and its elements. Therefore we avoid the use of the notion of a "network element". An analog discussion, yet about systems and subsystems, can you find in [122].

Let $(N^l)_{l \in L}$ be a family of networks $N^l = (G^l, V^l)$ with branch sets $Z^l$ and node sets $K^l$. If the networks of this family are not skeleton disjoint but fulfill the condition $sdV^k = sdV^l$ then it is always possible to adjoin to each of the networks $N^l$ an isomorphic network $\bar{N}^l$ such that these networks are skeleton disjoint and interconnectable. Such an approach is nowadays realized in every circuit simulator.

Definition II-E.20 Let $(N^l)_{l \in L}$ be the given family of intercon-

A quadruple $(N,H,A,k)$ is a network diagram if the following conditions are fulfilled:
Let \( N \) be a network with node set \( \mathcal{N} \) represented by the network diagram \((N,H,A)\) where \( H \subseteq E \times E \) is a hypergraph.

The behavior of a time-invariant network \( N \) is determined by \( \mathcal{A}(\mathcal{N}) \), the assignment of the \( \mathcal{K} \)-labeled \( \mathcal{K} \)-valued \( \mathcal{N} \)-dimensional time-invariant system which is defined by \( (\mathcal{G},\mathcal{P}) \) and \( \mathcal{G} \) is the \( \mathcal{K} \)-labeled \( \mathcal{N} \)-dimensional time-invariant system which is defined by \( (\mathcal{G},\mathcal{P}) \). The mapping \( \mathcal{G} \) is called a \( \mathcal{K} \)-labeled \( \mathcal{N} \)-dimensional time-invariant network.

The intersection \( \mathcal{K} \cap \text{dom} k \) is the terminal set of network \( \mathcal{N} \) (\( l \in L \)). The mapping \( \mathcal{G} \) realizes the assignment of the \( \mathcal{K} \)-labeled \( \mathcal{N} \)-dimensional time-invariant system which is defined by \( (\mathcal{G},\mathcal{P}) \).

If one agrees to use some graphical symbols to denote the networks which are to be connected, then the circuit diagrams used in colloquial engineering language can immediately be interpreted as graphic representations of circuit diagrams in the sense of Definition II-E.21. Of course, large networks can be represented by circuit diagrams, where some of their hyperbranches are itself represented by means of circuit diagrams. Clearly this process can be repeated several times. In that manner the gap between standard representations of networks and the network definition given above can be closed.

F. Classification of Networks and Elementary Properties of their Behaviors

Networks as introduced in Subsection II-B are defined as ordered pairs of a skeleton and a constitutive relation. Their definition is based on the spaces \( U, I, T, P, A \) and the bilinear mapping \( B \), introduced in Definition II-A.1. Therefore networks can be classified by means of the underlying 5-tuple \((U,I,T,P,A)\) and of the properties of their skeletons and their constitutive relations. Here only a few examples.

Definition II-F.1 \( \mathcal{N}(U,I,T,P,A) \) is the class of electrical networks if \( U := \mathbb{R}V, I := \mathbb{R}A, T := \mathbb{R}W, \mathcal{P} := \mathbb{R}, \mathcal{A} := \mathbb{R}, \mathcal{B} := \mathbb{R}, \mathcal{C} := \mathbb{R} \) and \( B \) is defined by \( \forall(U,I) \in U \times I \ B(U,I) = UI \).

\( \mathcal{N}(U,I,T,P,A) \) is the class of normalized networks if \( U := I := \mathbb{R}, T := \mathbb{R}W, \mathcal{P} := \mathbb{R}, \mathcal{A} := \mathbb{R} \) and \( B \) is defined by \( \forall(U,I) \in U \times I \ B(U,I) = UI \).

Definition II-F.2 Let \( \mathcal{N} \) be a network with skeleton \( C = (\mathcal{G}_v,\mathcal{G}_c) \).

This network has associated reference directions if \( \mathcal{G}_v = \mathcal{G}_c \).

It is a connected or unconnected network if its graphs \( \mathcal{G}_v \) and \( \mathcal{G}_c \) are both connected or unconnected, resp. □

The elementary networks introduced in Definition II-E.11 deliver another class of networks defined by properties of their skeletons.

Definition II-F.3 Let \( S \) be the universal signal set of some network and \( \mathcal{W} \subseteq S \) \( \forall T \in \mathbb{R} \) by \( \mathcal{W} = \{ (u(\tau),t) \mid (u(t),\tau) \in \mathcal{W} \} \).

\( \mathcal{W} \) is called time-invariant if \( T \in \mathbb{R} \) and \( \forall t \in T \{ x(t) \mid (x,t) \in \mathcal{W} \} = \mathcal{W} \).

Definition II-F.4 Let \( \mathcal{N} \) be a network with universal signal set \( S \) and constitutive relation \( V \).

\( \mathcal{N} \) is called time-invariant if its constitutive relation is time invariant. □

Proposition II-F.5 The behavior of a time-invariant network is time-invariant. □

In order to prepare the following definition, it is useful to recall some notions and facts of linear algebra [35]. \( V \) and \( W \) denote thereby two not necessarily finite dimensional linear spaces over the field of real numbers.

![Figure 7. Linear manifold and linear subspace of V](image)

A subset \( M \subseteq V \) is called a linear manifold of \( V \) if there exists a linear subspace \( U \) of \( V \) and an element \( a \in V \) such that \( M = \{ v + a \mid v \in U \} \).

Note, all linear subspaces of \( V \) are also linear manifolds.

Definition II-F.6 Let \( \mathcal{N} \) be a network with universal signal set \( S \) and constitutive relation \( V \).

\( \mathcal{N} \) is linear if each \( \forall T \in \mathbb{R} \) is a linear subspace of \( \mathcal{S}_T \).

\( \mathcal{N} \) is affine if each \( \forall T \in \mathbb{R} \) is a linear submanifold of \( \mathcal{S}_T \).

If \( \mathcal{N} \) is not linear, then it is called nonlinear. □

The fact that the intersection of linear subspaces of some vector space is always a linear subspace and that the intersection of linear manifolds is either a linear manifold or a void set implies the following theorem.

Theorem II-F.7 Let \( \mathcal{N} \) be a linear network with universal signal set \( S \) and solution set \( L \). Then \( \mathcal{L}_T \) is for each \( T \in \mathbb{R} \) a linear subspace of \( \mathcal{S}_T \). If \( \mathcal{N} \) is an affine network with a nonvoid solution set, then \( \mathcal{L}_T \) is for each \( T \in \mathbb{R} \) a linear submanifold of \( \mathcal{S}_T \). □

The Superposition Theorem follows immediately from Theorem II-F.7 [87]. The proof of generalizations of the
Thévenin-Norton Theorems presented in [87] is likewise based on Theorem II-F.7.

**Lemma and Definition II-F.8** If \( N \) is an affine network with constitutive relation \( V \), then there exists a signal pair \( (u^t, i^t) \in V \) with \( \text{dom } u^t = \bigcup_{T \in \mathcal{V}} T \). If \( V^\text{lin} \) is defined by \( V^\text{lin} := \{(u - u^t) | \text{dom } u , i - i^t | \text{dom } i \} \), \((u, i) \in V\), then the network \( N^\text{lin} := (C, V^\text{lin}) \) is a linear network and it holds \( V = \{(u + u^t) | \text{dom } u , i + i^t | \text{dom } i \} \). 

Under these assumptions the network \( N^\text{lin} \) is denoted as the linear network associated to the affine network \( N \). □

**Definition II-F.9** A network \( N = (C, V) \) with universal signal set \( S \) is denoted as resistive, if there exists a subset \( S' \subseteq S \) such that its constitutive relation can be reconstructed by means of \( S' \) from the family \( \{(V_i)_{i \in T} \) of its instantaneous value relations, i.e., if \( V = \{(u, i) \in S' | \forall t \in \text{dom } u (u(t), i(t)) \in V_i \} \).

Networks which are not resistive are denoted as dynamic. □

**Proposition II-F.10** Let \( N = (C, V) \) be a resistive network with universal signal set \( S \), let \( S' \subseteq S \) such that \( V = \{(u, i) \in S' | \forall t \in \text{dom } u (u(t), i(t)) \in V_i \} \) and let \( H \) be the Kirchhoff part of \( N \). Then \( L := \{(u, i) \in S' | \forall t \in \text{dom } u (u(t), i(t)) \in V_i \cap \mathcal{H} \} \) is the solution set of \( N \). □

Because of Proposition II-F.10 the determination of the solutions of a resistive network can be reduced to finite dimensional problems.

**Remark II-F.11** At this place it would be possible to introduce special kinds of networks such as independent voltage and current sources, open and short circuits, nullators and norators, fixators, resistors, controlled sources, gyrators, ideal transformers, inductors, coupled inductors, capacitors, memristors, etc. For details it is referred to [28], [16], [17], [19], [41]. □

The following definition is the base for a row of important tools for the development of network theory.

**Definition II-F.12** Let \( N \) and \( \tilde{N} \) be networks with the branch sets \( Z \) and \( \tilde{Z} \), resp., and the solution sets \( L \) and \( \tilde{L} \), resp. Let \( Z \subseteq \tilde{Z} \).

\( \tilde{N} \) generates the solutions of \( N \) if \( \tilde{L}_Z = L \). □

**Example II-F.13** We are given two electrical networks \( \tilde{N} \) and \( N \). Let \( \tilde{N} \) be a network as shown in Fig. 5 consisting of an ideal two-winding transformer with turns ratio 1 : 1 whose primary winding is terminated with an independent voltage source and whose secondary winding is terminated with a linear resistor, and let \( N \) be simply the parallel connection of voltage source and resistor from \( \tilde{N} \). Using suitable reference directions, suitable relative winding sense, and suitable signal sets for the networks \( N \) and \( \tilde{N} \) it is immediately to see that the “larger” network \( \tilde{N} \) generates the solutions of the “smaller” network \( N \). □

The network \( \tilde{N} \) sketched in the above example is a special case of a network in so called Belevitch normal form (for details and references cf. [89], [5]). It can be shown that for each network \( N \) there exists a network \( \tilde{N} \) in Belevitch normal form which generates the solutions of \( N \). The Paynter-networks introduced in [89] are even special cases of networks in Belevitch normal form. They can also be used to generate the solutions of a given network. As shown in [89], bondgraphs are simplified representations of Paynter-networks merely.

The following two theorems deliver likewise examples for Definition II-F.12. They are of central importance for the development of the theory of terminal behavior in Section III

**Theorem II-F.14** Let \( \tilde{N} = (\tilde{C}, \tilde{V}) \) be a network with branch set \( \tilde{Z} \). Then the sets \( Z^{oc} \) and \( Z^{sc} \) be disjoint subsets of \( Z \). The branch sets \( Z = \tilde{Z} \setminus (Z^{oc} \cup Z^{sc}) \), \( Z^{oc} \), and \( Z^{sc} \) are pairwise uncoupled and the subnetworks \( N_{Z^{oc}} \) and \( N_{Z^{sc}} \) may be consist exclusively of open-circuit or short-circuit branches, resp.

Let furthermore \( N = (C, V) \) be a network with branch set \( Z \) whose graphs are generated from that of \( \tilde{N} \) by deletion of the branches of \( Z^{oc} \) and contraction of the branches of \( Z^{sc} \).

If the constitutive relation of \( N \) obeys the condition \( V = \tilde{V}_Z \), then \( \tilde{N} \) generates the solution set of \( N \). □

A proof of Theorem II-F.14 can be given with tools developed in details in the Subsections 3.3 and 3.4 of [68] which are based on [105].

**Theorem II-F.15** Let \( N = (C, V) \) be a network with branch set \( Z \), node set \( K \) and solution set \( \tilde{L} \). The graphs of its skeleton are separabel and the node \( v_0 \in K \) be an articulation point of these graphs. Furthermore let \( N = (\tilde{C}, \tilde{V}) \) be a network with branch set \( \tilde{Z} = \tilde{Z} \), constitutive relation \( \tilde{V} = V \) and solution set \( \tilde{L} \), where the graphs of the skeleton of \( \tilde{N} \) are generated of that of \( \tilde{C} \) by means of a splitting of \( v_0 \). Then it holds the equation \( \tilde{L} = L \). □

A proof of this theorem follows immediately from the fact that each system of fundamental loops and fundamental cut sets of \( \tilde{N} \) is is also a system of fundamental loops and fundamental cut sets of \( \tilde{N} \) and vice versa. Therefore the Kirchhoff parts of the universal signal sets of these networks are equal and because of \( \tilde{V} = V \) it follows \( \tilde{L} = L \), too.

**Definition II-F.16** Let \( N \) be a network with branch set \( Z \) and universal signal set \( S \). Furthermore let \( Z^{ass} \) and \( Z^{opp} \) are the subsets of that branches of \( Z \) with associated or opposite reference directions, let \( (u, i) \in S \) and \( t \in \text{dom } u \).

If \( b \in Z^{ass} \), then the product \( u_b(t)i_b(t) \) is counted as the instantaneous power dissipated in branch \( b \) at time \( t \), and vice versa if \( b \in Z^{opp} \), then the product \( u_b(t)i_b(t) \) is counted as the instantaneous power delivered by branch \( b \) at time \( t \). □

The next theorem follows immediately from Kirchoff’s laws and the Orthogonality Theorem (cf. [104]) of graph theory, cf. [114], [102].

**Theorem II-F.17 (Theorem of Weyl-Tellegen)** Let \( N \) be a network with branch set \( Z \). Let \( H \) be the Kirchhoff part of the universal signal set of \( N \).

Then for each signal pair \( (u, i) \in H \) the corresponding instantaneous power values fulfill the equation \( \sum_{b \in Z^{ass}} u_b i_b = \sum_{b \in Z^{opp}} u_b i_b \). □
Since the solutions of a network has to fulfill Kirchhoff’s laws, the solutions of a network fulfill likewise such a balance condition.

**Corollary II-F.18** Let \( \mathcal{L} \) be the behavior of a network.

Then each signal pair \((u, i) \in \mathcal{L}\) fulfills the equation
\[
\sum_{b \in \mathcal{L}^{+}} u_{b} b = \sum_{b \in \mathcal{L}^{-}} u_{b} b . \quad \square
\]

The Weyl-Tellegen Theorem is one of the fundamental theorems of network theory. It delivers a physical interpretation of real and imaginary part of complex power [28] and is used for the proof of reciprocity theorems [28]. In particular together with the Colored Branch Lemma of G. Minty [61], [62], [111], [64] it is an important tool to prove theorems guaranteeing the uniqueness of the solutions of networks [121], [41], [40], [33]. We will give here only a short introduction to this field.

**Definition II-F.19** Let \( \mathcal{G} \) be an oriented graph with branch set \( \mathcal{Z} \).

A triple \((\mathcal{R}, \mathcal{G}, \mathcal{B})\) is denoted as a coloring of the branch set of \( \mathcal{G} \), if \( \mathcal{R}, \mathcal{G}, \mathcal{B} \) are pairwise disjoint subsets of \( \mathcal{Z} \), where at least one of these sets is nonvoid and their union is equal to the branch set of \( \mathcal{G} \). \( \square \)

In a mnemonic going back to G. Minty, the branches of the sets \( \mathcal{R}, \mathcal{G}, \mathcal{B} \) are denoted as colored red, green or blue.

**Lemma II-F.20 (Colored Branch Lemma)** Let \( \mathcal{G} \) be an oriented graph with branch set \( \mathcal{Z} \) and a coloring \((\mathcal{R}, \mathcal{G}, \mathcal{B})\) of \( \mathcal{Z} \). Let \( \mathcal{G} \) be nonvoid and \( g \) an element of \( \mathcal{G} \).

If \( \mathcal{E} \) denotes the set of all oriented loops and \( \mathcal{E} \) the set of all oriented cut sets of \( \mathcal{G} \), then exactly one of the following relationships

\[
\exists (z_{+}, z_{-}) \in \mathcal{E} \quad (g \in \mathcal{Z}^{+} \subseteq \mathcal{G} \cup \mathcal{Z}^{-} \subseteq \mathcal{R})
\]

or

\[
\exists (z_{+}, z_{-}) \in \mathcal{E} \quad (g \in \mathcal{Z}^{+} \subseteq \mathcal{G} \cup \mathcal{B} \cup \mathcal{Z}^{-} \subseteq \mathcal{B})
\]

is fulfilled. \( \square \)

That means, the graph \( \mathcal{G} \) includes under the given assumptions either at least one oriented loop of green or red colored branches or at least one oriented cut set of green and blue colored branches, where the green colored branches of such a loop or cut set are similarly oriented to the prescribed branch \( g \in \mathcal{G} \).

**Theorem II-F.21** Let \( \mathcal{N} \) be an affine resistive network with associated reference directions which consists of positive resistors without couplings and independent voltage and current sources.

If \( \mathcal{N} \) includes no loops of independent voltage sources and no cut sets of independent current sources, then it has on each interval of time axis a unique solution.

For a proof of Theorem II-F.21 let \( z \) denote the number of branches of \( \mathcal{N} \). Then the constitutive relation of \( \mathcal{N} \) can be represented by a system of inhomogenous linear algebraic equations with \( z \) linearly independent variables. Kirchhoff’s laws lead to a system of homogeneous linear algebraic equations likewise with \( z \) linearly independent equations and \( 2z \) variables. The solutions of \( \mathcal{N} \) have to fulfill the resulting inhomogenous system of \( 2z \) equations for \( 2z \) unknown quantities which is therefore a system of behavioral equations of \( \mathcal{N} \).

It is well-known from linear algebra, such a system of inhomogenous linear algebraic equations has a unique solution if and only if the associated homogeneous system has only a trivial solution.

The corresponding homogeneous system of the behavioral equations of \( \mathcal{N} \) can be interpreted as a system of behavioral equations of a network \( \mathcal{N} \) which is built from \( \mathcal{N} \) if all voltage sources of \( \mathcal{N} \) are replaced by short circuit branches and all current sources by open circuit branches. In that manner, the branch set of \( \mathcal{N} \) is partitioned into three disjoint subsets \( \mathcal{Z}_{R}, \mathcal{Z}_{S} \) and \( \mathcal{Z}_{O} \). The set \( \mathcal{Z}_{R} \) includes the branches of the resistors of \( \mathcal{N} \). The set \( \mathcal{Z}_{S} \) includes the short circuit branches and \( \mathcal{Z}_{O} \) the open circuit branches of \( \mathcal{N} \). It is not necessary that all these sets are nonvoid.

By assumption \( \mathcal{N} \) does not include any loops or cut sets consisting exclusively of branches of \( \mathcal{Z}_{S} \) or \( \mathcal{Z}_{O} \), resp.

Obviously, these assumptions are necessary therefore that \( \mathcal{N} \) can have on each interval of the time axis only a trivial solution since otherwise, without a contradiction to Kirchhoff’s laws and the constitutive equations of \( \mathcal{N} \), in a loop of short circuit branches could be circulate an arbitrary loop current and to the branch voltages of a cut set of open circuit branches could be an arbitrary cut set voltage added.

To show that these assumptions are also sufficient therefore that \( \mathcal{N} \) can have on each interval of the time axis only a trivial solution we start from any solution \((u, i)\) of \( \mathcal{N} \) and show that this solution fulfills necessarily the condition \((u, i) = (0, 0)\).

The Weyl-Tellegen Theorem implies
\[
\sum_{b \in \mathcal{Z}_{R}} u_{b} i_{b} + \sum_{b \in \mathcal{Z}_{S}} u_{b} i_{b} + \sum_{b \in \mathcal{Z}_{O}} u_{b} i_{b} = 0 . \quad (*)
\]

From the constitutive equations of \( \mathcal{N} \) it follows
\[
u_{\mathcal{Z}_{S}} = 0 \quad \text{and} \quad i_{\mathcal{Z}_{O}} = 0 ,
\]
i.e. \( \forall b \in \mathcal{Z} \left( b \in \mathcal{Z}_{S} \Rightarrow u_{b} = 0 \right) \) and \( \forall b \in \mathcal{Z} \left( b \in \mathcal{Z}_{O} \Rightarrow i_{b} = 0 \right) \). Therefore the second and the third term on the left hand side of Equation \((*)\) are equal to zero.

Let \((R_{b})_{b \in \mathcal{Z}_{R}}\) be the family of the resistances of the branches \( b \in \mathcal{Z}_{R} \). These resistances are positive by assumption. Because \( u_{b} = R_{b} i_{b} \) it follows \( u_{b} i_{b} = R_{b} (i_{b})^{2} \geq 0 \) for \( i_{b} \neq 0 \). Together with \( u_{\mathcal{Z}_{S}} = 0 \) and \( i_{\mathcal{Z}_{O}} = 0 \) the identity \((*)\) delivers
\[
\sum_{b \in \mathcal{Z}_{R}} R_{b} (i_{b})^{2} = 0 .
\]

Therefore it is
\[
u_{\mathcal{Z}_{R}} = 0 \quad \text{and} \quad i_{\mathcal{Z}_{R}} = 0 .
\]

Now we must show that
\[
u_{\mathcal{Z}_{S}} = 0 \quad \text{and} \quad u_{\mathcal{Z}_{O}} = 0 .
\]
are true.
If \( Z_S = \emptyset \) or \( Z_O = \emptyset \), then the corresponding relationship 
\[ \forall b \in Z \ (b \in \emptyset \Rightarrow u_b = 0) \] 
and 
\[ \forall b \in Z \ (b \in \emptyset \Rightarrow i_b = 0) \]

is already fulfilled and there is nothing to prove.

The remaining nontrivial cases can be treated with the help of the Colored Branch Lemma.

If \( Z_S \neq \emptyset \), then we choose an arbitrary branch from \( Z_S \) and denote it \( g_1 \). This branch is now colored “green”. The remaining branches of \( Z_S \) are colored “red” and that of \( Z_R \cup Z_O \) are colored “blue”. It results in \( \mathfrak{G}_1 := \{g_1\}, \mathfrak{R}_1 := Z_S \setminus \{g_1\} \) and \( \mathfrak{B} := Z_R \cup Z_O \).

Because there does not exist any loop including branch \( g_1 \) together with branches of \( \mathfrak{R}_1 \) there exists a cut set including \( g_1 \) and possibly additional branches of \( \mathfrak{B} \). Since \( i_{g_1} = 0 \) it follows \( i_{g_1} = 0 \) from the corresponding cut set equation.

If \( \mathfrak{R}_1 \neq \emptyset \), then we choose an arbitrary branch from \( \mathfrak{R}_1 \) and denote it \( g_2 \). This branch is now colored “green”. The remaining branches of \( \mathfrak{R}_1 \) are left “red” and that of \( Z_R \cup Z_O \) are left unchanged “blue”, resulting in \( \mathfrak{G}_2 := \{g_1, g_2\}, \mathfrak{R}_2 := \mathfrak{R}_1 \setminus \{g_2\} \) and \( \mathfrak{B} := Z_R \cup Z_O \), and so further and so on.

This construction is stepwise to repeat for all branches of \( Z_S \). The procedure stops if all branches of \( Z_S \) are colored “green” and the equation \( i_{Z_S} = 0 \) is proved.

If \( Z_O \neq \emptyset \), then we choose an arbitrary branch from \( Z_O \) and denote it \( g_1 \). This branch is now colored “green”. The remaining branches of \( Z_O \) are colored “blue” and that of \( Z_R \cup Z_S \) are colored “red”. It results in \( \mathfrak{G}_1 := \{g_1\}, \mathfrak{B}_1 := Z_O \setminus \{g_1\} \) and \( \mathfrak{R} := Z_R \cup Z_S \).

Because there does not exist any cut set including branch \( g_1 \) together with branches of \( \mathfrak{B}_1 \) there exists a loop including \( g_1 \) and possibly additional branches of \( \mathfrak{R} \). Since \( u_{g_1} = 0 \) it follows \( u_{g_1} = 0 \) from the corresponding loop equation.

This construction is stepwise to repeat for all branches of \( Z_O \). The procedure stops if all branches of \( Z_O \) are colored “green” and the equation \( u_{Z_O} = 0 \) is proved. \( \Box \)

Remark II-F.22 The proof given above can be immediately generalized to the class of affine resistive networks, at which the constitutive relation of their linear subnetworks can be given with matrix representation, characterized by a constitutive equation of the form \( u_R(t) = R i_R(t) \), where \( R \) is a positive definite matrix, which must not be a diagonal matrix. It means, then, that there are no couplings of the resistors allowed, yet not of that of controlled sources.

The proof of the above version of the theorem can be transferred to that class of RLC networks which are linear and time-invariant with exception of independent voltage and current sources and do not include closely coupled inductors. For this purpose we consider their behavioral equations in Laplace-domain. If we restrict the complex frequency variable \( p = \sigma + j \omega \) to some positive real value \( \sigma_0 \), then these restricted behavioral equations can be interpreted as behavioral equations of a resistive network which is linear with exception of independent sources and it follows that the determinant of the behavioral equations in Laplace-domain cannot vanish identically, cf. [99], p. 139.

For additional generalizations to RLC networks with coupled branches we refer to [60], [78].

A cornerstone of the proof of Theorem II-F.21 is the observation that the power consumed in a resistor branch is positive if its branch current does not vanish. The subsequent steps of the above proof can be used to prove that a resistive network \( N = (C, V) \) consisting of independent voltage and current sources and nonlinear uncoupled resistors with strict monotonically increasing instantaneous value characteristics \( V_b \in \mathcal{U} \times \mathcal{I} \) can have on each interval of the time axis at most one solution.

For an indirect proof we assume the existence of two different solutions \( (u, i) \) and \( (\bar{u}, \bar{i}) \). Since they are solutions of \( N \) they satisfy Kirchhoff’s laws. Their difference \( \Delta u, \Delta i \) := \((u - \bar{u}, i - \bar{i})\) satisfies Kirchhoff’s laws for a modified network, where the voltage sources are replaced by short circuit branches and the current sources by open circuit branches. Then it follows from the Weyl-Tellegen Theorem \( \sum_{b \in Z_R} (\Delta u)_b (\Delta i)_b = 0 \), where \( Z_R \) denotes the branch set of its nonlinear resistors with strictly monotone instantaneous value characteristics. From the monotonicity of the resistor characteristics it follows that \( (\Delta u)_b (\Delta i)_b > 0 \) for \( b \in Z_R \). In the same manner as in the proof of Theorem II-F.21 it follows that \( \Delta u = 0 \) and \( \Delta i = 0 \). That means, the network \( N \) can have on each interval of the time axis at most only one solution, cf. [41], [29], [40]. For additional results we refer to [121]. \( \Box \)

The proof of the so-called no-gain property of resistive networks is also based on the Colored Branch Lemma [111], [120].

Remark II-F.23 For the description of concrete examples it is sometimes useful to assign to a network with branch set \( Z \) the families \( (U_b)_{b \in Z} \) and \( (I_b)_{b \in Z} \) defined for all \( b \in Z \) by \( U_b := \mathcal{U} \) and \( I_b := \mathcal{I} \), resp.

Then the following identities \( \mathcal{U}^Z := \prod_{b \in Z} U_b = \{(U_b)_{b \in Z} \mid \forall b \in Z U_b_i \in U_b\} \) and \( \mathcal{I}^Z := \prod_{b \in Z} I_b = \{(I_b)_{b \in Z} \mid \forall b \in Z I_b_i \in I_b\} \) hold.

The mappings \( p_{V_b} \) and \( p_{C_b} \) introduced in Definition II-B.3 are the corresponding canonical projections of these products. \( \Box \)

Theorem II-F.24 (Covering Theorem) Let \( N = (C, V) \) be a network with universal signal set \( S \), Kirchhoff part \( H \) and behavior \( L := V \cap H \). Let furthermore \( (V^l)_{l \in L} \) be a family of not necessarily disjoint subsets of \( V \) with \( V = \bigcup_{l \in L} V^l \) and let \( (L^l)_{l \in L} \) be the family of sets defined by \( L^l := V^l \cap H \). Then \( L = \bigcup_{l \in L} L^l \).

If all \( V^l \) \((l \in L)\) are restriction compatible, then all ordered pairs \( N^l := (C, V^l) \) are networks. \( \Box \)

The proof of Theorem II-F.24 follows from elementary rules of set algebra. Although its proof is very simple, it has turned out to be very useful, both for the completion of network theory and for its applications [36], [37], [38], [91], [25], [26], [67], [66].

Remark II-F.25 Let \( N = (C, V) \) be a network, then the family \( (V_T)_{T \in dL} \) delivers a simple example of such a covering. By means of this covering it is possible to prescribe the domain of the solutions of \( N \). Obviously, the corresponding networks \( N_T := (C, V_T) \) are then trivial restriction compatible.
By means of \( L := \{(U, I, t_0, T) \in U^2 \times I^2 \times T \times \text{Int} T \mid t_0 \in T \} \) and \( V(U, I, t_0, T) := \{(u, i) \in V \mid \text{dom} u = T \land u(t_0) = U \land i(t_0) = I \} \) it is possible to prescribe domain and initial value of the solutions of \( N \). Analogously zero-state initial values can be prescribed.

Using the Covering Theorem it is, in many cases, possible to reduce the determination of the constitutive relation of a canonical representative (cf. Definition III-B.20 and Theorem III-B.22 in Section III more below) of the terminal behavior of a multipole to standard network analysis problems with unique solutions. The essential idea thereby is the use of coverings of the constitutive relations of the norators mentioned in Theorem III-B.22 by means of families of constitutive relations of voltage sources, series connections of such sources with linear resistors, current sources or fixators.

\[ \begin{align*}
\text{Figure 8.} & \quad \text{Coverings of the Instantaneous-Value Relation of a Norator, subsets of typical coverings of } U \times I \\
\text{Notations II-F.1} & \quad N = (C, V) \text{ and } \bar{N} = (C, \bar{V}) \\
\text{Theorem II-F.26 (Intersection Theorem)} & \quad \text{If there exists a subset } W \subset S \text{ with } \bar{V} = V \cap W, \text{ then } \bar{N} \text{ is a network with behavior } \bar{L} = L \cap W. \quad \Box
\end{align*} \]

\[ \begin{align*}
\text{Figure 9.} & \quad \text{Illustration of the Inclusion Lemma by a Venn diagram} \\
\text{Theorem II-F.28 (Invariance Theorem)} & \quad L = \bar{L} \text{ if and only if } L \subseteq \bar{V} \text{ and } \bar{L} \subseteq V.
\end{align*} \]

\[ \begin{align*}
\text{Figure 10.} & \quad \text{Illustration of the Invariance Theorem by a Venn diagram} \\
\text{Theorem II-F.29 (General Substitution Theorem)} & \quad \text{If } L \subseteq \bar{V} \text{ and if additionally the solution sets } L \text{ and } \bar{L} \text{ have the same finite number of elements, then the networks } N \text{ and } \bar{N} \text{ have the same solution sets, i.e., } L = \bar{L}. \quad \Box
\end{align*} \]

The proof of the assertion of the Inclusion Lemma follows then from \( L \subseteq \bar{V} \) by an intersection with \( H \). It delivers \( L \cap H \subseteq \bar{V} \cap H \). The right hand side of this relationship is equal to \( \bar{L} \). Because of \( H \cap H = H \) and \( L \cap H = (V \cap H) \cap H = V \cap (H \cap H) = L \) the left hand side of this relationship is equal to \( L \), i.e., \( L \subseteq \bar{L} \). \( \Box \)

To verify the condition \( L \subseteq \bar{V} \wedge \bar{L} \subseteq V \) we need in general both \( L \) and \( \bar{L} \). The next theorem shows that the condition \( L = \bar{L} \) can be guaranteed by means of a weaker condition. Its proof follows immediately from fundamental properties of finite sets and the set of natural numbers.

\[ \begin{align*}
\text{Theorem II-F.30} & \quad \text{Let } N \text{ be a network with a finite solution}
\end{align*} \]
set and let $\mathcal{Z}' \subset \mathcal{Z}$ be a subset of the branch set of $\mathcal{N}$ such that $\forall (u, i) \in \mathcal{L} \ i \mathcal{Z}' = 0$.

Furthermore, let the constitutive relation of $\mathcal{N}$ fulfill the condition $\mathcal{V} = \{(u, i) \in \mathcal{V} \mid i \mathcal{Z}' = 0\}$.

If the solution sets of $\mathcal{N}$ and $\mathcal{N}'$ have the same finite number of elements, then it is $\mathcal{L} = \mathcal{L}'$.

If $(u, i) \in \mathcal{L}$, then by assumption it is $i \mathcal{Z}' = 0$. This implies the relationship $(u, i) \in \mathcal{V}$, i.e., $\mathcal{L} \subseteq \mathcal{V}$.

Note, for the proof of Corollary II-F.30 it is not necessary that the complementary subsets $\mathcal{Z}'$ and $\mathcal{Z}'' := \mathcal{Z} \setminus \mathcal{Z}'$ of $\mathcal{Z}$ are uncoupled. For instance, $\mathcal{N}$ can include a current controlled voltage source $\mathcal{N}_{\{a,b\}}$ as a subnetwork, where the branch voltage of $b \in \mathcal{Z}''$ is controlled by the branch current of $a \in \mathcal{Z}'$. Note also, that in general $\mathcal{V} \neq \mathcal{V}'$.

**Example II-F.31** Fig. 11 shows two linear resistive networks, $\mathcal{N}_\alpha$ and $\mathcal{N}$. It may assumed that they have both associated reference directions. It may further assumed that $R_1 = R_2 = \ldots = R_5 > 0$.

The system of constitutive equations of $\mathcal{N}_\alpha$ includes a parameter $\alpha \in \mathbb{R}$. For $\alpha = 0$ this network is equal to $\mathcal{N}$ which consists only of positive uncoupled resistors exited by a voltage source whose branch is not a selfloop. According to Theorem II-F.21 $\mathcal{N}$ has a unique solution. Since the determinants of arbitrary behavioral equations of $\mathcal{N}_\alpha$ are continuous functions of $\alpha$, the network $\mathcal{N}_\alpha$ has at least for sufficiently small values of $|\alpha|$ a unique solution. Moreover, since such a determinant is an affine function of $\alpha$ it follows that at most only one exceptional point $\alpha_0$ exists at which such a determinant vanishes. Obviously, for each $\alpha \neq \alpha_0$ the solution of $\mathcal{N}_\alpha$ is uniquely determined by $u_1 = u_2 = u_3 = u_4 = \frac{5}{4} u_7$, $u_5 = 0$, $i_5 = 0$ and $u_6 = 0$. That means, because $i_5 = 0$ the network $\mathcal{N}_\alpha$ can be for all $\alpha \neq \alpha_0$ replaced by $\mathcal{N}$. □

**Theorem II-F.32** The networks $\mathcal{N}$ and $\mathcal{N}'$ have the same solution sets if the following conditions are fulfilled:

(i) $\mathcal{L}$ and $\mathcal{L}'$ have the same finite number of elements.

(ii) It exists a partition $(\mathcal{Z}', \mathcal{Z}'')$ of the branch set of $\mathcal{N}$ and $\mathcal{N}'$ such that $\mathcal{Z}'$ and $\mathcal{Z}''$ are uncoupled as well in $\mathcal{N}$ as in $\mathcal{N}'$.

(iii) $\mathcal{L}_{\mathcal{Z}'} \subseteq \mathcal{V}_{\mathcal{Z}'}$.

(iv) $\mathcal{V}_{\mathcal{Z}''} = \mathcal{V}_{\mathcal{Z}'''}$.

$\mathcal{Z}'$ and $\mathcal{Z}''$ are uncoupled in $\mathcal{N}$ and $\mathcal{N}'$ if and only if $\mathcal{V} = \mathcal{V}_{\mathcal{Z}'} \otimes \mathcal{V}_{\mathcal{Z}''}$ and $\mathcal{V} = \mathcal{V}_{\mathcal{Z}'} \otimes \mathcal{V}_{\mathcal{Z}'''}$. Obviously, $\mathcal{L} \subseteq \mathcal{V}$ implies $\mathcal{L}_{\mathcal{Z}'} \subseteq \mathcal{V}_{\mathcal{Z}'}$ and $\mathcal{L}_{\mathcal{Z}''} \subseteq \mathcal{V}_{\mathcal{Z}''}$. By assumption it is $\mathcal{L}_{\mathcal{Z}'} \subseteq \mathcal{V}_{\mathcal{Z}'}$ and $\mathcal{V}_{\mathcal{Z}''} = \mathcal{V}_{\mathcal{Z}'''}$. Therefore it holds $\mathcal{L}_{\mathcal{Z}'} \subseteq \mathcal{V}_{\mathcal{Z}'}$ and $\mathcal{L}_{\mathcal{Z}''} \subseteq \mathcal{V}_{\mathcal{Z}'''}$, which implies $\mathcal{L} \subseteq \mathcal{V}_{\mathcal{Z}'} \otimes \mathcal{V}_{\mathcal{Z}''}$ and thereby $\mathcal{L} \subseteq \mathcal{V}$. Because $|\mathcal{L}| = |\mathcal{L}'| < N_0$ it is then $\mathcal{L} = \mathcal{L}'$. □

**Theorem II-F.33** Let $\mathcal{N}$ have a unique solution and let $(\mathcal{Z}', \mathcal{Z}'')$ be a partition of the branch set of $\mathcal{N}$ in uncoupled subsets.

Then there exists a modified network $\mathcal{N}'$ with the following properties:

(i) $(\mathcal{Z}', \mathcal{Z}'')$ is also a partition of the branch set of $\mathcal{N}'$ into uncoupled subsets,

(ii) $\mathcal{N}_{\mathcal{Z}'}$ is a nondegenerated affine resistive network with $\mathcal{V}_{\mathcal{Z}'} \supseteq \mathcal{L}_{\mathcal{Z}'}$,

(iii) $\mathcal{V}_{\mathcal{Z}''} = \mathcal{V}_{\mathcal{Z}'''}$.

If $\mathcal{N}$ has also a unique solution, then it is equal to the solution of $\mathcal{N}'$.

It is always possible to choose $\mathcal{N}_{\mathcal{Z}'}$ in such a manner that its associated linear network is time-invariant.

**Proof:** Since $\mathcal{N}$ and $\mathcal{N}'$ have unique solutions it is possible to assume that their constitutive relations are trivial restriction compatible. If $(\hat{u}, \hat{i})$ denotes the unique solution of $\mathcal{N}$, then the condition $\mathcal{L}_{\mathcal{Z}'} \subseteq \mathcal{V}_{\mathcal{Z}'}$ is equivalent to $(\hat{u}_{\mathcal{Z}'}, \hat{i}_{\mathcal{Z}'}) \in \mathcal{V}_{\mathcal{Z}'}$. The instantaneous value relations of a nondegenerated resistive network with $z' := |\mathcal{Z}'|$ branches are $z'$-dimensional linear submanifolds of the configuration space of such a network. If we identify the signal pairs $(u', i') \in \mathcal{V}_{\mathcal{Z}'}$ by some numbering of the branches of $\mathcal{Z}'$ with column matrices, then the ansatz $A(u' - \hat{u}_{\mathcal{Z}'}) + B(i' - \hat{i}_{\mathcal{Z}'}) = 0$ delivers for each pair $(A, B)$ of real $z' \times z'$-matrices with $\text{rank}(A, B) = z'$ a constitutive equation which defines a constitutive relation fulfilling the prescribed conditions.

Obviously, the equation $Au' + Bi' = 0$ defines the constitutive relation of a time-invariant linear resistive network.

Because of $|\mathcal{L}| = |\mathcal{L}'| = 1$ and $\mathcal{V}_{\mathcal{Z}''} \supseteq \mathcal{V}_{\mathcal{Z}'''}$ it is $\mathcal{L} = \mathcal{L}'$. □

**Remark II-F.34** From Theorem II-F.33 one obtains as well the Classical Substitution Theorem as also a row of variants of Generalized Substitution Theorems. But because results are rarely known which guarantee for a nonlinear network the existence of exactly $n \geq 2$ solutions, one is usually forced to replace the condition $|\mathcal{L}| = |\mathcal{L}'| < N_0$ by $|\mathcal{L}| = |\mathcal{L}'| = 1$. 

![Network Diagram](image-url)
The following list includes some Variants of Generalized Substitution Theorems:

1°) The Classical Substitution Theorem, cf. [28], is that special case of Theorem II-F.33, at which \( Z' \) includes only one branch and \( \mathcal{N}_{Z'} \) consists either of an independent voltage or an independent current source.

2°) \( \mathcal{N}_{Z'} \) is a nondegenerated affine resistive network with only one branch which can be equivalently replaced either by a series connection of a linear resistor and an independent voltage source or by a parallel connection of a linear resistor and an independent current source.

3°) \( |Z'| \geq 2, \mathcal{N}_{Z'} \) consists exclusively of independent voltage and independent current sources.

4°) \( |Z'| \geq 2, \mathcal{N}_{Z'} \) consists exclusively of independent voltage sources, independent current sources, fixators and norators, where the number of fixators is equal to the number of norators.

5°) \( |Z'| = 2n, \mathcal{N}_{Z'} \) consists of \( n \) fixators and \( n \) norators. □

![Figure 12. Analysis of a ladder network by means of Classical Substitution Theorem](image)

The analysis of RLC ladder networks delivers nice examples for an application as well of the Classical Substitution Theorem as for an application of some of its generalizations.

**Example II-F.35** The transfer functions of linear ladder networks can be determined in frequency domain by a recursive procedure which is based on a repeated application of voltage or current divider rule and a representation of their input impedance by means of a continued fraction [28], p. 298 of the impedances of their branches. For this approach it is not necessary to solve any linear algebraic equations.

![Figure 13. Analysis of a ladder network by means of a generalized substitution and the Superposition Theorem](image)

For convenience we sketch here only the computation of the output voltage \( u_8 \) of the resistive ladder network shown in Fig. 12 a). In a first step we determine the input resistance \( R_{E_1} \) of the dashed bordered subnetwork shown in Fig. 12 b, c). A resistor with this resistance is the canonical representative (cf. Definition III-B.20) of the terminal behavior of this subnetwork. Therefore the dashed bordered subnetwork can be replaced by this resistor. The resulting network is shown in Fig. 12 d). The voltage divider rule delivers now the voltage \( u_2 \). Because \( R_{E_1} \) represents the terminal behavior of the dashed bordered subnetwork shown in Fig. 12 b) this voltage is indeed equal to the voltage \( u_2 \) in Fig. 12 e). Now the dashed bordered subnetwork in Fig. 12 e) is not replaced by its Thévenin-Norton equivalent network but rather by a canonical representative of its terminal behavior. This canonical representative is an affine resistive two-pole. The resulting network is shown in Fig. 12 f). Since we know its branch voltage, we can use the Classical Substitution Theorem to replace this affine resistor by an independent voltage source with the prescribed voltage \( u_2 \). Now we can repeat this procedure for the “smaller” ladder network shown in Fig. 12 g) and in a finite number of steps this algorithm delivers the output voltage of the given ladder network.

If all the resistors of the given ladder network are positive then Theorem II-F.21 assures that all considered networks have on a prescribed interval of the time axis a unique solution and Remark II-F.22 guarantees this for RLC ladder networks, too.

In that manner it is indeed possible to determine the output voltage of a ladder network without solving of linear algebraic equations. Yet, the application of the Superposition Theorem and a generalized substitution theorem leads to a much more effective algorithm.

For this purpose it is only necessary to replace the voltage source in the input branch of a given RLC ladder network, cf. Fig. 13 a), by a norator and the output open circuit branch by a fixator, cf. Fig. 13 b). While the prescribed current of this fixator is equal to zero it is chosen at first an arbitrary value for the phasor of its prescribed voltage. The analysis of the so modified network leads in frequency domain with the help of a system loop-current and node-voltage equations.
together with a suitable ordering of these equations to a system of inhomogeneous linear algebraic equations whose coefficient matrix is an upper triangular matrix with all diagonal elements equal to 1. This fact guarantees for the modified network the existence of a unique solution. The existence of a unique solution for the originally given RLC ladder network follows from Theorem II-F.21 together with Remark II-F.22 if all its resistors, capacitors and inductors are positive.

Obviously, because of the special structure of this network the computation of the solution of the modified ladder network can be carried out without to set up any equations from its network diagram.

The same idea underlies the construction of phasor diagrams for linear networks of series-parallel type.

In a last step the phasor of all branch voltages and currents determined in that manner for the modified network are multiplied with a suitable complex number to match the voltage of the norator with the originally given value of the input voltage. □

Without the network theoretical shimming, given here, this kind of equations was already considered in [8].

This approach for the adjustment of DC-operating points of transistor circuits proposed in [109] (213 – 217) (cf. also [110], p. 234 – 237) as an example without an explanation of its network theoretical background. It is systematically used in [57]. Obviously, it can be immediately generalized to circuits including more then only one transistor or other three- or multipole devices.

With this approach it is also possible to prove the formulae collected in Table 8.1 in [15], p. 380.

For further examples and applications of the Covering and the General Substitution Theorem we refer to [28], [36], [38], [82], [90], [65].

III. Terminal Behavior of Networks

A. Multipoles and Multiports

In applications, a given physical device can be connected with other such objects only on its terminals at which it is accessible from “outside”. If these physical devices are modeled by networks, then some distinguished subsets of their node sets take on the role of the terminals.

Often it is useful, sometimes even necessary, to partition the terminal set of a device into pairwise disjoint subsets. Typical examples are partitions into sets of input and output terminals of transmission networks, such as filters, amplifiers and transmission lines [52], pp. 580 – 585, 619 – 646. Other examples are starlike interconnections of a “central” with its “satellites”.

Such considerations motivate the subsequent definitions.
Definition III-A.1 Let \( \mathcal{N} \) be a network. Then the ordered pair \((\mathcal{N}, K)\) is denoted as a multipole if \( K = (K^l)_{l \in L} \) is a family of pairwise disjoint subsets of the node set of \( \mathcal{N} \). The network \( \mathcal{N} \) is the underlying network of \((\mathcal{N}, K)\). The family \( K \) is its terminal class family. The sets \( K^l \) \((l \in L)\) are called the terminal classes of \((\mathcal{N}, K)\) and \( K^{ts} := \bigcup_{l \in L} K^l \) is its terminal set.

A multipole \((\mathcal{N}, K)\) is denoted as a multiport if each of its terminal classes includes exactly two elements. The terminal classes of a multiport are denoted as terminal pairs. \( \square \)

![Figure 15. Multipole \((\mathcal{N}, K)\) with terminal classes \(K^1, K^2\) and \(K^3\)](image)

Notations III-A.1 Let \((\mathcal{N}, K)\) be a multipole and let \( n := (n^l)_{l \in L} \) be the family defined by \( n^l := |K^l| \) \((l \in L)\), then this multipole is sometimes also denoted as an \( n \)-pole. If the family \( K = (K^l)_{l \in L} \) includes only one terminal class, then this multipole is denoted for convenience also by \((\mathcal{N}, K^{ts})\). In this case the multipole \((\mathcal{N}, K^{ts})\) is denoted as an \( n \)-pole, where \( n \) is now defined by \( n := |K^{ts}| \).

Analogously, a multiport with \( n \) terminal pairs is also denoted as an \( n \)-port. \( \square \)

Definition III-A.2 A multipole is denoted as an elementary multipole if the graphs of its underlying network are forests such that the node set of each component of such a forest is equal to exactly one of its terminal classes.

A multiport is denoted as an elementary multiport if each component of the graphs of its underlying network consists of only one branch connecting the elements of the corresponding terminal pair. \( \square \)

Definition III-A.3 Let \((\mathcal{N}, K)\) be a multipole or a multiport and let \( \mathcal{N}' \) be an external network interconnectable with \( \mathcal{N} \). An interconnection of \( \mathcal{N} \) and \( \mathcal{N}' \) is called a terminal-class conform interconnection if some of the nodes of \( \mathcal{N}' \) are identified with terminals of \((\mathcal{N}, K)\) such that terminals of different terminal classes belong to different components of \( \mathcal{N}' \). \( \square \)

Observe, it is not necessary that all terminals of some terminal class of \((\mathcal{N}, K)\) belong in a terminal-class conform interconnection of \( \mathcal{N} \) and \( \mathcal{N}' \) to the same component of \( \mathcal{N}' \). Observe also, that it is allowed that branches of different components of \( \mathcal{N}' \) are coupled with each other.

The notion of a multipole with a family of disjoint terminal classes introduced in Definition III-A.1 is closely related to the notion of a multi-sectional multipole introduced by H. MANN [55], [56].

As we will shown in the next subsection it is possible to replace each multipole by an elementary multipole with the same terminal set and the same terminal behavior with respect to terminal-class conform interconnections with external networks. Analogously it exists to each multiport in the sense of Definition III-A.1 an elementary multiport with the same family of terminal pairs and the same terminal behavior with respect to terminal-pair conform interconnections with external networks. Such elementary multipoles and multiports are in the following denoted as the canonical representatives of the given multipole or multiport. Yet, to prove such a theorem it is necessary that the class of all admitted networks is sufficiently general. For instance, a canonical representative of RLC multipole is in general not included in the class of RLC networks.

The notion of a multiport goes back to the theory of microwave circuits [63], [115], [9]. However, up to the present day it is not used in a unique manner. Already in [47] it was suggested to admit instead of terminal pairs also terminal multi-tuples with more than two terminals as ports. Relatively often one meets also the opinion that each multipole with \( n + 1 \) terminals would be always an \( n \)-port. Sometimes this version is additionally connected with the condition that the terminal behavior of such an \( n \)-port can be represented by means of a system of \( n \) independent equations. Then nullators, fixators, norators and some voltage and current mirrors, as introduced in [3], would be from the outset excluded. In particular the last mentioned restriction is inappropriate for the development of a sufficiently general theory.

B. Labelled Networks

Let \((\mathcal{N}, K)\) be a given multipole. The terminal behavior of \((\mathcal{N}, K)\) has to characterize the influence of \( \mathcal{N} \) on arbitrary external networks \( \mathcal{N}' \) connected at the terminals of \((\mathcal{N}, K)\) in such a manner that after a replacement of the given multipole \((\mathcal{N}, K)\) in such an interconnection by another one, \((\mathcal{N}', K)\), with the same terminal behavior we are unable to observe this exchange in any of the external networks \( \mathcal{N}' \) which can be connected as well with the terminals of \((\mathcal{N}, K)\) as with that of \((\mathcal{N}, K)\).

In the majority of practical applications it may be sufficient to regard only multipoles with given terminal classes. However, representation and development of the theory of terminal behavior is substantially simplified if one starts, in accordance
with a long standing tradition in electrical engineering (thing on the +, − labels for the terminals of a battery, etc.), from the more flexible class of labelled networks. On demand, it is then at any time possible to restrict the achieved results to the special case of labelled networks whose label functions are identical embeddings.

**Definition III-B.1** A label class family is a finite family of pairwise disjoint finite nonvoid sets denoted as label classes.

If $M := (M^l)_{l \in L}$ is a label class family, then $UM := \bigcup_{l \in L} M^l$ is the label set of $M$.

The cases $|L| = 1$ and $\exists_{l \in L} |M^l| = 1$ are admitted.

**Definition III-B.2** A triple $(N, M, \mu)$ is called an $M$-labelled network, an $M$-network or (short) a labelled network if $N$ is a network, $M$ a label class family and $\mu$ an injective mapping of $UM$ into the node set of $N$. The map $\mu$ is the label function of this labelled network. $N$ is denoted as the underlying network of $(N, M, \mu)$.

**Proposition III-B.3** Let $(N, M, \mu)$ be a labelled network. Then the sets $K^l := \mu(M^l)$ $(l \in L)$ are pairwise disjoint. If $K$ denotes the family $(K^l)_{l \in L}$ of these sets, then $(N, K)$ is a multipole with $K$ as its terminal class family.

Vice versa, if $(N, K)$ is a multipole with terminal class family $K = (K^l)_{l \in L}$, then a labelled network $(N', M, \mu')$ is assigned to this multipole in a unique manner by means of $M' := K^l$ $(l \in L)$. $M := (M^l)_{l \in L}$ and $\mu := \text{id}_{UM} := \{(m, m) \mid m \in UM\}$.

**Definition III-B.4** Let $M$ be a label class family. An $M$-labelled network $(N, M, \mu)$ is component conform $M$-labelled if terminals of different terminal classes belong to different components of $N$, too.

In analogy to Definition III-A.3 it is in Definition III-B.4 not supposed that all terminals of some terminal class of a component conform $M$-labelled network belong to the same component of this network.

**Definition III-B.5** Let $M := (M^l)_{l \in L}$ be a prescribed label class family and $S_*$ a prescribed signal type.

Then $\mathfrak{M}_M$ denotes in the following the class of all $M$-labelled networks whose constitutive relations are included in $S_*$-signal sets.

$\mathcal{E}_M \subset \mathfrak{M}_M$ denotes the subset of all $M$-labelled networks of $\mathfrak{M}_M$ whose label functions are identical embeddings.

$\mathcal{C}_M \subset \mathfrak{M}_M$ denotes the subset of all component conform $M$-labelled networks of $\mathfrak{M}_M$.

$\tilde{\mathcal{F}}_M \subset \mathcal{E}_M$ denotes the subset of all $M$-networks of $\mathcal{E}_M$ whose label functions are bijections, whose graphs are forests and whose terminal classes are the node sets of the trees of these forests.

$\mathfrak{D}_M(N) \subset \mathcal{E}_M$ denotes the set of all $M$-networks of $\mathcal{E}_M$ which are skeleton disjoint to a given labelled network $N \in \mathfrak{M}_M$.

$\mathfrak{D}_M(N) \subset \tilde{\mathcal{F}}_M$ denotes the set of all norator networks of $\tilde{\mathcal{F}}_M$ which are skeleton disjoint to a given labelled network $N \in \mathfrak{M}_M$.

For convenience, letters as $N, \tilde{N}, \hat{N}, \check{N}$, etc. are also used in the following as variables to denote elements of $\mathfrak{M}_M$.

The sets $\mathfrak{D}_M(N)$ and $\mathfrak{D}_M(N)$ deliver the test networks for the determination of the terminal behavior of a given $M$-labelled network $N$. There is no loss of generality if we confine yourself to this class of networks and do not use the subset of all that networks of $\mathcal{E}_M$ which are interconnectable with $N$ as test networks since even to each network $N' \in \mathcal{E}_M$ there exists a network $N' \in \mathfrak{D}_M(N)$ which is isomorphic to $N'$.

**Definition III-B.6** Let $N, \tilde{N} \in \mathfrak{M}_M$ and let $\tilde{N} \in \mathfrak{D}_M(N)$, where $\mu, \bar{\mu}$ and $\tilde{\mu}$ denote the label function of $N, \tilde{N}$ and $\hat{N}$, resp.

The network $\tilde{N}$ is denoted as a direct $M$-interconnection of $N$ and $\bar{N}$ if this interconnection is generated by a function $\kappa$ which maps for each $m \in UM$ the terminals $\mu(m)$ and $\tilde{\mu}(m)$ to $\tilde{\mu}(m)$ and all nodes which are not terminals of $N$ and $\bar{N}$ are identified by $\kappa$ with itself.

For the definition of the terminal behavior of a network it is useful to introduce besides of the direct interconnection of skeleton-disjoint labelled networks an additional kind of interconnections of such networks.

**Definition III-B.7** Let $N, \tilde{N} \in \mathfrak{M}_M$ and let $\tilde{N} \in \mathfrak{D}_M(N)$, where $\mu, \bar{\mu}$, and $\tilde{\mu}$ denote the label function of $N, \tilde{N}$, and $\hat{N}$, resp.

The network $\tilde{N}$ is denoted as an extended $M$-interconnection of $N$ and $\bar{N}$ if it fulfills the following conditions:

(i) $\tilde{N}$ can be decomposed into four subnetworks. These subnetworks are, besides of $N$ and $\bar{N}$, two other networks, denoted by $\mathcal{N}_V$ and $\mathcal{N}_A$, consisting exclusively of open or short circuits, resp.

(ii) The voltage and current graph of $\mathcal{N}_A$ consists of $|UM|$ components. Each of these components consists of a series connection of two branches connecting a terminal of $\hat{N}$ with that one of $\tilde{N}$ at which the same label is stuck. For each $m \in UM$ the terminal $\bar{\mu}(m)$ of $\tilde{N}$ is the internal node of that component of $\mathcal{N}_A$ which connects the terminals $\mu(m)$ and $\tilde{\mu}(m)$. The branches of voltage and current graph of $\mathcal{N}_A$ are oriented towards $\bar{\mu}(m)$.

(iii) The network $\mathcal{N}_V$ consists of $|L|$ components. Each of these components has one of the sets $\mu(M^l)$ $(l \in L)$ as
its node set. In that component of the voltage and current graph of \( N_N \) with the node set \( \tilde{\mu}(M) \) there exist for each terminal pair \((\hat{\mu}(m), \hat{\mu}(n))\) \((m,n \in M, m \neq n)\) exactly two branches connecting these terminals. One of these two branches is oriented from \( \hat{\mu}(m) \) to \( \hat{\mu}(n) \) the other one from \( \hat{\mu}(n) \) to \( \hat{\mu}(m) \). □
dom$\bar{u}$ and $\forall_{t \in \text{dom} u} u(t) = \bar{u}(t) \circ \zeta_{N^-,\hat{N}} \land i(t) = \bar{i}(t) \circ \xi_{N^-,\hat{N}}$. □

**Definition III-B.16** Let $M(N, \hat{N})$ denote the complete image of the behavior of the extended interconnection $\hat{N} = E_M(N, \hat{N})$ of $N$ and $\hat{N}$ under the projection $P_{N,\hat{N}}$. □

Now we are ready to define the notion of terminal behavior.

**Definition III-B.17** Let $N \in \Omega_M$. Then the set $V_M(N) := \bigcup_{\hat{N} \in \partial_M(N)} L_M(N, \hat{N})$ is denoted as the $M$-terminal behavior of $N$.

Obviously, $V_M(N) \subseteq S_M$ holds for all $N \in \Omega_M$.

**Definition III-B.18** The symbol $\simeq_M$ denotes the equivalence relation defined for $N, \hat{N} \in \Omega_M$ by means of $N \simeq_M \hat{N} \iff V_M(N) = V_M(\hat{N})$.

Labelled networks $N$ and $\hat{N}$ with the property $N \simeq_M \hat{N}$ are denoted as $M$-equivalent.

$[N]_M := \{N \in \Omega_M \mid N \simeq_M N\}$ is the class of all networks which are $M$-equivalent to $N$.

The elements of $[N]_M$ are the representatives of the $M$-terminal behavior of $N$. □

**Theorem III-B.19** (Main Theorem) Let $N^1$ and $N^2$ be $M$-equivalent networks of $\Omega_M$. Let $N \in (\partial_M(N^1) \cap \partial_M(N^2))$ be a labelled network with branch set $Z$. If $L^1$ and $L^2$ are the solution sets of the extended $M$-interconnections of $N^1$ and $N^2$ with $N$, resp., then the relationship $L^1_Z = L^2_Z$ holds.

If $L^1$ and $L^2$ are the solution sets of the direct $M$-interconnections of $N^1$ and $N^2$ with $N$, resp., then the relationship $L^1_Z = L^2_Z$ holds.

For a proof of this theorem it must be shown that the solution sets of the extended $M$-interconnections of $N$ with $N^1$ and $N^2$ obey the relationships $L^1_Z \subseteq L^2_Z$ and $L^1_Z \supseteq L^2_Z$.

Let us start with a verification of the inclusion $L^1_Z \subseteq L^2_Z$.

If $(\bar{u}, \bar{i}) \in L^1_Z$, then there exists a signal pair $(\bar{u}^1, \bar{i}^1) \in L^1$ with $(\bar{u}, \bar{i}) = (\bar{u}^1, \bar{i}^1)$ and an $M$-signal pair $(\bar{u}, \bar{i}) \in S_M$ with $(\bar{u}, \bar{i}) = P_{N^1,X}(\bar{u}^1, \bar{i}^1)$. All these signal pairs have the same domain, $T := \text{dom} \bar{u}$.

It is now shown that there exists a solution $(\bar{u}^2, \bar{i}^2) \in L^2_Z$ of the extended interconnection of $N^2$ with $N$ obeying the conditions $(\bar{u}^2, \bar{i}^2) = (\bar{u}, \bar{i})$ and therefore $P_{N^2,X}(\bar{u}^2, \bar{i}^2) = (\bar{u}, \bar{i})$.

Since $N^1$ and $N^2$ have the same $M$-terminal behavior there exists at least a network $N \in \partial_M(N^2)$ such that the solution set $\hat{L}^2$ of the extended interconnection of $N^2$ with $N$ includes a signal pair $(\bar{u}^2, \bar{i}^2)$ obeying the condition $P_{N^2,X}(\bar{u}^2, \bar{i}^2) = (\bar{u}, \bar{i})$.

All in all, we have to consider three different extended interconnections. That are $\hat{N}^1 := E_M(N^1, \hat{N})$, $\hat{N}^2 := E_M(N^2, \hat{N})$ and $\hat{N}^\alpha := E_M(N^\alpha, \hat{N})$. The volt- and ampermeter networks of $\hat{N}^1$, $\hat{N}^2$ and $\hat{N}^\alpha$ have in general different branch and node sets but their graphs are isomorphic. The corresponding bijections between their branch and node sets are determined together with the label functions of the networks $N^1$, $N^2$ and $N^\alpha$ and that ones of $\hat{N}^1$, $\hat{N}^2$ and $\hat{N}^\alpha$ by means of the mappings $\zeta_{N^1,\hat{N}}, \zeta_{N^2,\hat{N}}, \zeta_{N^\alpha,\hat{N}}, \zeta_{\hat{N}^1,\hat{N}}, \zeta_{\hat{N}^2,\hat{N}}, \zeta_{\hat{N}^\alpha,\hat{N}}$.

Because of $P_{N^1,\hat{N}}(\bar{u}^1, \bar{i}^1) = P_{N^2,\hat{N}}(\bar{u}^2, \bar{i}^2) = (\bar{u}, \bar{i})$ the branch voltages and branch currents of corresponding branches in the volt- and ampermeter networks of $\hat{N}^1$ and $\hat{N}^\alpha$ are equal. The wanted solution $(\bar{u}^2, \bar{i}^2)$ of $N^2$ can be therefore combined from components of the solutions $(\bar{u}^1, \bar{i}^1)$ of $N^1$ and $(\bar{u}^\alpha, \bar{i}^\alpha)$ of $N^\alpha$. If the branch sets of the volt- and ampermeter networks of $N^1$ are denoted by $Z^1$ and $Z^\alpha$, resp., and that one of his subnetworks $N$ and $N^2$ by means of $Z$ and $Z^2$, resp., then $(\bar{u}^2, \bar{i}^2)$ has to obey the conditions

$\bar{u}^2_Z = \bar{u}^1_Z, \quad \bar{u}^2_Z = 0, \quad \bar{u}^2_Z = \bar{u}^\alpha_Z, \quad \forall_{t \in T} \left( \bar{u}^2_Z(t) = \bar{u}(t) \circ (\zeta_{N^2,\hat{N}})^{-1} \right)$,

and

$\bar{Z} = \bar{Z}^1, \quad \bar{i}^2_Z = 0, \quad \bar{i}^2_Z = \bar{i}^\alpha_Z, \quad \forall_{t \in T} \left( \bar{i}^2_Z(t) = \bar{i}(t) \circ (\zeta_{N^2,\hat{N}})^{-1} \right)$.

Since the projections of $(\bar{u}^2, \bar{i}^2)$ defined by the branch sets $Z$, $Z^1$, $Z^\alpha$ and $Z^2$ are itself projections of solutions of some networks, they obey the constitutive relations of that subnetworks of $N^2$ generated by these branch sets. Therefore $(\bar{u}^2, \bar{i}^2)$ satisfies the constitutive relation of $N^2$, too.

In order to show that $\bar{u}^2$ satisfies Kirchhoff’s voltage law it is sufficient to show that $\bar{u}^2$ satisfies the loop equations of system of fundamental loops of the voltage graph of $N^2$. For this purpose we choose at first in the voltage graph of $N^2$ a complete forest. Then each component of this forest is completed by means of exactly one series connection of branches of the voltage graph of $N^2$. The graph defined in that manner is loopless and can therefore extended to a complete forest of $N^2$. This complete forest of the voltage graph of $N^2$ determines a system of fundamental loops. Each of these fundamental loops includes either exclusively branches of $Z^1 \cup Z^\alpha \cup Z$ or of $Z^2 \cup Z^1 \cup Z^\alpha$.

The voltages $\bar{u}^1$ and $\bar{u}^\alpha$ across the branch sets of $\hat{N}^1$ and $\hat{N}^\alpha$, resp., satisfy the loop equations corresponding to branches of $Z$ or $Z^2$ together with some branches of the volt- and ampermeter networks of $N^1$ or $N^\alpha$, resp. Because of the isomorphisms between the graphs of the different volt- and ampermeter networks of $\hat{N}^1$, $\hat{N}^\alpha$, and $N^2$, mentioned above, they satisfy also the loop equations of the just now introduced system of fundamental loops of $N^2$.

The currents $\bar{i}^1$ and $\bar{i}^\alpha$ through the branch sets of $\hat{N}^1$ and $\hat{N}^\alpha$, resp. satisfy the node equations corresponding to the internal nodes of $N$ and $N^2$. But they fulfill also the node equations at their terminals and at the nodes of the corresponding voltmeter networks. Likewise because of the isomorphisms between the graphs of the different volt-
amperemeter networks of \( \tilde{N}^1 \), \( \tilde{N} \) and \( N^2 \), mentioned above, the satisfy the node equations of \( N^2 \).

Thereby the inclusion \( \tilde{L}_2^1 \subseteq \tilde{L}_2^2 \) is proved.

With analogue reasonings it can be shown that \( (\tilde{u}^2, \tilde{i}^2) \in \tilde{L}_2^2 \) implies \( (\tilde{u}^2, \tilde{i}^2) \in \tilde{L}_2^1 \).

The second statement of the theorem to be proved follows from the fact that an extended \( M \)-interconnection of \( N^1 \), respectively of \( N^2 \), with \( N \) generates the solutions of the direct interconnections of these networks.

This theorem, which we call the Main Theorem of the theory of terminal behavior, asserts that there does not exist any network \( N \in \mathcal{L}_M \) which is interconnectable both with \( N^1 \) and \( N^2 \) in which we are able to decide whether \( N \) is connected to \( N^1 \) or to \( N^2 \). With other words, this theorem shows that our definition meets the essential aspect of the intuitive notion of the terminal behavior of a network.

**Definition III-B.20** The class \( \mathfrak{E}_M(N) := \mathfrak{g}_M \cap [N]_M \) is the class of all canonical representatives of the terminal behavior \( N \in \mathfrak{T}_M \). \( \Box \)

The canonical representatives are networks without loops, i.e., their underlying networks are elementary networks.

Let \( \hat{N} \) be a canonical representative of the terminal behavior of some labelled network \( N \). Using Kirchhoff’s laws it is then possible to determine all \( M \)-signal pairs of \( \mathcal{V}_M(N) \) and vice versa. With other words, each canonical representative of an \( M \)-network delivers an equivalent representation of its \( M \)-terminal behavior.

By definition it is \( \mathfrak{M} \mathfrak{R}_M(N) \subseteq \mathfrak{S}_M(N) \). Likewise by definition exists to each \( M \)-signal pair \( (u, i) \in \mathcal{V}_M(N) \) a labelled network \( N \in \mathfrak{S}_M(N) \) such that \( (u, i) \in \mathfrak{L}_M(N, \bar{N}) \). Obviously, this \( M \)-signal pair \( (u, i) \) is also consistent with an extended interconnection of \( N \) with any norator network \( \hat{N} \in \mathfrak{M} \mathfrak{R}_M(N) \), i.e., \( (u, i) \in \mathfrak{L}_M(N, \bar{N}) \).

Therefore it is for the determination of \( \mathcal{V}_M(N) \), i.e., of \( \bigcup_{N \in \mathfrak{S}_M(N)} \mathfrak{L}_M(N, \bar{N}) \), not necessary to determine the solution sets of all extended \( M \)-interconnections of the given network \( N \) with the uncountably infinite set of networks \( \hat{N} \in \mathfrak{S}_M(N) \). But rather it is possible to restrict us to only one interconnection of \( N \) with a norator forest of \( \mathfrak{M} \mathfrak{R}_M(N) \).

**Theorem III-B.21** Let \( N \in \mathfrak{T}_M \) be a given network and \( \hat{N} \) an arbitrary norator network \( \hat{N} \in \mathfrak{M} \mathfrak{R}_M(N) \). Then \( \mathcal{V}_M(N) = \mathfrak{L}_M(N, \bar{N}) \). \( \Box \)

Moreover, one receives a canonical representative of a given labelled network immediately from its interconnection with a norator forest of \( \mathfrak{M} \mathfrak{R}_M(N) \).

---

**Theorem III-B.22** Let \( N \in \mathfrak{T}_M \) be a given \( M \)-network and \( \hat{N} \in \mathfrak{M} \mathfrak{R}_M(N) \). Let \( \mathcal{L} \) denote the solution set of the direct interconnection of \( N \) and \( \bar{N} \). Let \( \mathcal{G} := (\mathcal{G}_v, \mathcal{G}_c) \) be the skeleton of \( N \) and \( \bar{\mu} \) its label function. Furthermore let \(-\mathcal{G}_c \) denote the oppositely oriented graph to \( \mathcal{G}_c \).

Then the \( M \)-network \( \hat{N} := ((\mathcal{L}, \mathcal{V}), M, \bar{\mu}) \) with the skeleton \( \mathcal{G} := (\mathcal{G}_c, -\mathcal{G}_c) \) and the voltage-current relation \( \mathcal{V} := \mathcal{L}_2 \) is a canonical representative of the \( M \)-terminal behavior of the given \( M \)-network. \( \Box \)
behavior. This method can immediately be transferred to the case of linear time-invariant lumped networks using frequency-domain methods or the elimination algorithm of DERVISOGLOU and DESOER [27].

\[
\begin{align*}
&\begin{array}{c}
A \\
B \\
\end{array} \quad \begin{array}{c}
C \\
D \\
\end{array} \\
\end{align*}
\begin{array}{c}
\simeq_M \\
\simeq_M \\
\end{array}
\begin{array}{c}
A \\
B \\
\end{array} \\
\begin{array}{c}
C \\
D \\
\end{array}
\begin{array}{c}
i_1 \\
i_2 = i_1 \\
\end{array}
\begin{array}{c}
u_2 = u_2 \\
u_3 = u_2 \\
\end{array}
\end{align*}
\]

Figure 21. Realization of a nullor by means of controlled sources

Example III-B.23

Fig. 21 shows for the label class family \( M \) defined by \( L := \{1\} \) and \( M^1 := \{A, B, C, D\} \) two \( M \)-equivalent resistive \( M \)-networks of \( \mathcal{E}_M \). Their \( M \)-equivalence can be proved by solving the corresponding behavioral equations for the determination of their canonical representatives by means of Gauss elimination. \( \square \)

\[
\begin{align*}
&\begin{array}{c}
A \\
B \\
\end{array} \quad \begin{array}{c}
A \\
B \\
\end{array} \\
\end{align*}
\begin{array}{c}
\simeq_M \\
\simeq_M \\
\end{array}
\begin{array}{c}
A \\
B \\
\end{array} \\
\begin{array}{c}
A \\
B \\
\end{array}
\]

Figure 22. Realizations of a nullor and a norator by means of controlled sources

Example III-B.24

Fig. 22 and Fig. 23 show for the label class family \( M \) defined by \( L := \{1\} \) and \( M^1 := \{A, B\} \) pairs of \( M \)-equivalent resistive \( M \)-networks of \( \mathcal{E}_M \). Their \( M \)-equivalence can likewise be proved by solving the corresponding behavioral equations for the determination of their canonical representatives by means of Gauss elimination. \( \square \)

\[
\begin{align*}
&\begin{array}{c}
A \\
B \\
\end{array} \quad \begin{array}{c}
A \\
B \\
\end{array} \\
\end{align*}
\begin{array}{c}
\simeq_M \\
\simeq_M \\
\end{array}
\begin{array}{c}
A \\
B \\
\end{array} \\
\begin{array}{c}
A \\
B \\
\end{array}
\]

Figure 23. Realization of a fixator by means of a nullator and independent sources [101]

In contrast to most of the textbooks the Examples III-B.23 and III-B.24 show that that one must permit nullors, nullators, norators and fixators, if one accepts controlled sources. Example III-B.23 shows furthermore how it is possible to describe nullors by means of the input language of standard circuit analysis programs.

Example III-B.25

Fig. 24 shows for the label class family \( M \) defined by \( L := \{1\} \) and \( M^1 := \{A, B, C\} \) two \( M \)-equivalent linear resistive \( M \)-networks \( \mathcal{N}, \mathcal{N}' \in \mathcal{E}_M \). We assume that their underlying networks have associated reference directions. The underlying network of \( \mathcal{N} \) consists of a linear resistor with conductance \( G \) and an isolated node. Its constitutive relation is defined by the equation \( i = Gu \). The \( M \)-network \( \mathcal{N} \) is a canonical representative of the \( M \)-terminal behavior of \( \mathcal{N} \). Its underlying network consists of a two coupled linear resistors. Its constitutive relation is defined by the equation

\[
\begin{pmatrix}
i_1 \\
i_2
\end{pmatrix} =
\begin{pmatrix}
G & -G \\
-G & G
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}.
\]

\( \square \)

For nonlinear resistive networks these theorems open additional “geometrical” insight into the structure of their driving point and transfer characteristics (cf. [80], [91]).

With the Covering Theorem of network analysis (Theorem II-F.24) it is possible to develop further variants (cf. [82], [91]) from the last mentioned theorems which allow the computation of the terminal behavior of a network \( \mathcal{N} \in \mathcal{Y}_M \) by means of a “small” subset of \( \mathcal{S}\mathcal{D}_M\mathcal{N} \) which is “easy” to survey in comparison to the set \( \mathcal{S}\mathcal{D}_M\mathcal{N} \) itself. Well-known examples are the methods for the computation of impedance, admittance, hybrid, chain, and scattering representations of the terminal behavior of linear networks. If these networks are time-invariant and lumped then the algorithm of [27] yields constructive realizations.

Beyond that, the Theorems III-B.21 and III-B.22 supply efficient principles for the proof of many other theorems on the terminal behavior of networks. In that manner one receives clear proofs as well as for the Theorems of HELMHOLTZ and MAYER as for the theorems of CHUA and GREEN [20] on the elimination from \( C \)-loops and \( L \)-cut sets, cf. [83], [85], [86]. An additional advantage of these proofs is the fact that they get along without recourse to the Substitution Theorem with the uniqueness conditions which are then to attend.

In [50] a special algorithm for the computation of multiple operating points of nonlinear resistive networks was suggested. The mathematical analysis of this algorithm, given in [91], is also based on the theory of terminal behavior. Our proof [84] of the auxiliary branch method is likewise based on this theory (cf. [46], [32], [103], [92]).

It is pointed out here that the notion of a multipole with several terminal classes was for the proof of the Theorem of CHUA and GREEN on the elimination of cut sets consisting of inductances and independent current sources presented in [85] substantially used. Also some transmission line models lead inevitably to multipoles with several terminal classes, cf. [52].
C. Properties of the Terminal Behavior of Labelled Networks

Notations III-C.1 In the following denotes \( M = (M^l)_{l \in L} \) a label class family. As in Subsection III-B \( V = (V^l)_{l \in L} \) denotes the family of index sets assigned to \((M^l)_{l \in L}\). \( U M \) denotes the union of the \( M^l \) and \( U V \) the union of the \( V^l \) (\( l \in L \)). \( \square \)

Proposition and Definition III-C.1 Let the map \( \text{id}_{UV} : UV \to UM \times UM \), defined by the assignment \((m,n) \mapsto (m,n)\), be the identical embedding of \( V \subset UM \times UM \) into \( UM \times UM \).

Then the triple \( G_M := (UV, UM, \text{id}_V) \) is an oriented graph.

This graph is denoted as an \( M \)-graph. \( \square \)

![Figure 25. Example of an \( M \)-graph for \( M := (M^l)_{l \in L}, L := \{1\} \) and \( M^1 := \{A,B,C,D\} \)](image)

Obviously, this graph is isomorphic to the graphs of the voltmeter subnetwork of the extended \( M \)-interconnection of arbitrary networks \( N \in \mathfrak{N}_M \) and \( \hat{N} \in \mathfrak{S}_M(\hat{N}) \).

Several properties of the terminal behavior of labelled networks can be clearly formulated with the help of this kind of graphs.

Definition III-C.2 A subset \( \mathcal{W} \subset \mathcal{S}_M \) is denoted as an \( M \)\-relation if the following conditions hold:

(i) \( \mathcal{W} \) is restriction compatible.

(ii) Each \( M \)\-signal pair \((u,i) \in \mathcal{S}_M \) fulfills for each oriented mesh \((Z^+,Z^-)\) of \( G_M \) the equation \( \sum_{v \in Z^+} u_v - \sum_{v \in Z^-} u_v = 0 \).

(iii) Each \( M \)\-signal pair \((u,i) \in \mathcal{S}_M \) fulfills for each \( m \in UM \) the equation \( i_{(m,+1)} = -i_{(m,-1)} \) and for each \( l \in L \) the equation \( \sum_{m \in M^l} i_{(m,+1)} = 0 \). \( \square \)

Since the \( M \)-terminal behavior of a network \( \hat{N} \in \mathfrak{N}_M \) can be computed as a projection of the solution set of its extended \( M \)-interconnection with a norator network \( \hat{N} \in \mathfrak{S}_M(\hat{N}) \) it holds the following theorem.

**Theorem III-C.3** For each network \( \hat{N} \in \mathfrak{N}_M \) the \( M \)-terminal behavior is an \( M \)-relation. \( \square \)

The next theorem shows that our formalization of network theory is closed in respect to the notion of terminal behavior. Moreover it delivers an impetus for the modeling of real physical devices by means of multipoles.

**Theorem III-C.4** Let \( \mathcal{W} \subset \mathcal{S}_M \) and let \( \mathcal{W} \) additionally be an \( M \)-relation.

Then there exists a labelled network \( \hat{N} \in \mathfrak{N}_M \) such that its terminal behavior is equal to \( \mathcal{W} \).

Its proof is very simple. Starting with the \( M \)-relation \( \mathcal{W} \) we can immediately construct the constitutive relation of a network \( \hat{N} \in \mathfrak{S}_M \) with \( \mathcal{V}_M(\hat{N}) = \mathcal{W} \). Its equivalence class \([\hat{N}]_M \) includes the set of all labelled networks with this terminal behavior. \( \square \)

In addition it follows from this theorem that to a real technical device exists an always network model, if the behavior of this device can be described at least at its terminals sufficiently exact in terms of voltages and currents.

Together with the possibility to represent large networks by means of hierarchically nested interconnections of smaller networks this theorem supplies the basis for the development of network-theoretical models for the simulation of large technical circuits, cf. [119].

Clearly, these remarks guarantee only the existence of network-theoretical device models. They deliver neither any hint to find in the equivalence class \([\hat{N}]_M \), mentioned just now, an appropriate element, nor they solve the problem to find a representation of the constitutive relation of such a model by means of behavioral equations and they do not give any support, in order to find all equivalent models, etc.

The theory of transmission lines delivers typical examples at which network-theoretical models are able to represent only the terminal behavior of such a device [106], [107], [108], [70], [52].

As it was said already further above, the terminal behavior of a labelled network \( \hat{N} \) can be represented not only by the \( M \)-relation \( \mathcal{V}_M(\hat{N}) \), but also by each canonical representative \( \hat{N} \in \mathfrak{S}_M(\hat{N}) \) of its terminal behavior.

Additionally to these representations now still a representation of the terminal behavior of labelled networks by terminal potentials and terminal currents will be introduced.

**Definition III-C.5** Let \( M \) be a label class family.

Then \( \mathcal{S}_M^\phi \) is the set defined by \( \mathcal{S}_M^\phi := \{ (u^\phi,v^\phi) \in \bigcup_{T \in \text{Int_T}} (\mathcal{U}^{M^T})^T \times (\mathcal{U}^{M^T})^T | \forall m \in UM \; (u^\phi_m,v^\phi_m) \in S_t \} \), where the time-functions \( u^\phi_m \) and \( v^\phi_m \) are defined for all \((u^\phi,v^\phi) \in \mathcal{S}_M^\phi, m \in UM \) and \( t \in \text{dom} \; u^\phi = \text{dom} \; v^\phi \) by \( u^\phi_m(t) := u^\phi(t)(m) \) and \( v^\phi_m(t) := v^\phi(t)(m) \), resp. \( \square \)

**Lemma and Definition III-C.6** The assignment \( \mathcal{W} \mapsto \mathcal{W}^\phi := \{ (u^\phi,v^\phi) \in \mathcal{S}_M^\phi | \exists (u,i) \in \mathcal{W} \; u^\phi_m - u^\phi_n = u_{(m,n)}^i \wedge i^\phi_m = i_{(m,-1)}^i \} \) defines a map of the set of all \( M \)-relations \( \mathcal{W} \in \mathfrak{P}(\mathcal{S}_M) \) into the set \( \mathfrak{P}(\mathcal{S}_M^\phi) \).

\( \mathcal{W}^\phi \) is denoted as the potential representation of \( \mathcal{W} \). \( \square \)

**Definition III-C.7** If \( \hat{N} \) is a labelled network, then the potential representation of its terminal behavior \( \mathcal{V}_M(\hat{N}) \) is denoted by \( \mathcal{V}_M^\phi(\hat{N}) \). \( \square \)

The potential representation of the terminal behavior of labelled networks is a generalization of the frequency-domain
representation of the terminal behavior of a linear, time-invariant multipole by an indefinite admittance matrix [10].

**Remark III-C.8** From the potential representation of the terminal behavior of a multipole one receives in a simple manner those canonical representatives of its terminal behavior, whose graphs are forests consisting of star shaped trees, where the node set of each of these trees is exactly one of the terminal classes of the multipole under consideration. For this purpose in each terminal class an element is selected as a reference terminal. This terminal is then used as the center of the corresponding tree. The terminal currents of the remaining terminals of the corresponding terminal class and the potential differences between these terminals and the selected reference terminal deliver the constitutive relation of such a canonical representative. □

Graph-theoretical results (cf. [6], Theorem 2, p. 149), yield a proof of the next theorem.

**Theorem III-C.9** A subset \( \mathcal{W} \subseteq S_M \) is an \( M \)-relation, if and only if its potential representation \( \mathcal{W}^φ \) fulfills the conditions:

(i) \( \mathcal{W}^φ \) is restriction compatible,
(ii) \( \forall (u^φ, v^φ) \in S^φ_M \) \( \left[ (u^φ, v^φ) \in \mathcal{W}^φ \land \forall l \in L \forall m, n \in M^l \right] \)
\( v^φ_m = v^φ_n \) \( \Rightarrow \) \( (u^φ + v^φ, i^φ) \in \mathcal{W}^φ \),
(iii) \( \forall (u^φ, i^φ) \in \mathcal{W}^φ \forall l \in L \sum_{n \in M^l} i^φ_n = 0 \). □

This statement is indeed of fundamental importance for the development of network theory, since it allows a comparison of different axiomatic approaches of network theory. It states that the defining conditions of the notion of an \( M \)-relation are equivalent to that formulated in [31], [48], [41], [19], [86] as introductory axioms for the definition of multipoles. It implies also that the capability of the class of networks introduced in Section II suffices to construct network-theoretical models for technical devices, of course under the assumption, mentioned already more above, that their terminal behavior can be described in terms of voltages and currents, cf. the considerations in the Concluding Remarks of [86].

**Remark III-C.10** The determination of the potential representation of the terminal behavior of a labelled network can be likewise reduced to an analysis of an interconnection of its terminals with a norator forest. Thereby each terminal of the given network is connected with one of the poles of a norator 2-pole where the other poles of these norators are identified separately according to their corresponding terminal classes in additionally external nodes. □

Now we would like to examine non-terminal-class-conform interconnections of multipoles. Cause for these considerations were some problems of classical 4-pole theory. A simple example of this kind concerns the relationships between four-poles, three-poles and two-ports.

At first, we consider the question under which assumptions it is possible to infer from the \( M \)-terminal behavior of a labelled network to its terminal behavior with respect to so-called subdivisions of \( M \).

**Definition III-C.11** Let \( M = (M^l)_{l \in L} \) and \( N = (N^j)_{j \in J} \) be two given label class families.

Then the label class family \( N \) is a subdivision of \( M \) if it fulfills the conditions:

(i) \( \bigcup_{l \in L} M^l = \bigcup_{j \in J} N^j \).

(ii) There exists a partition \( (J^l)_{l \in L} \) of the index set of \( N \) into pairwise disjoint nonvoid subsets such that \( \forall_{l \in L} M^l = \bigcup_{j \in J^l} N^j \). □

**Theorem III-C.12** Let \( M \) be a label class family and \( (N, M, \mu) \) be an \( N \)-network of \( \mathfrak{N}_M \). Furthermore let \( N \) be a subdivision of \( M \) and \( \mathcal{C} \) be the skeleton of an \( N \)-network from \( \mathfrak{N}_N \).

Then there exists a canonical representative \( (\tilde{N}, M, \mu) \) of the \( M \)-terminal behavior of \( (N, M, \mu) \) whose skeleton \( \mathcal{C} \) includes the graphs of \( \mathcal{C} \) as subgraphs.

If \( \tilde{V} \) denotes the constitutive relation of \( \tilde{N} \) and \( \tilde{Z} \) and \( \tilde{Z} \) denote the branch sets of \( \mathcal{C} \) and \( \mathcal{C} \), resp., then the \( N \)-network \( (\tilde{N}, N, \mu) \) defined by
\[
\tilde{V} := \{ (\tilde{u}, i) \mid \exists_{(u, i) \in V} \tilde{u} = \tilde{u}_{\mathcal{C}} \land i = i_{\mathcal{C}} \land i_{\mathcal{C} \setminus \tilde{Z}} = 0 \}
\]

and
\[
\tilde{N} := (\tilde{C}, \tilde{V})
\]
is a canonical representative of the \( N \)-terminal behavior of \( (N, N, \mu) \). □

By the way, the representation of \( \tilde{V} \) in Theorem III-C.12 is an example of a representation of a constitutive relation using a latent quantity in the sense of [118], [77], [119]. The latent quantity is here the partial voltage across \( \tilde{Z} \setminus \tilde{Z} \).

The difficulties of the classical four-pole theory (anyway an inappropriate notation for a two-port theory) are connected with the reverse task, the question whether the description of the \( N \)-terminal behavior of an \( N \)-network \( (N, N, \mu) \in \mathfrak{N}_N \) delivers also a description of the \( M \)-terminal behavior of the labelled network \( (N, M, \mu) \in \mathfrak{N}_M \), if \( N \) is a subdivision of \( M \).

For the construction of the constitutive relation of the network \( N \) described in Theorem III-C.12 the condition \( i_{\mathcal{C} \setminus \tilde{Z}} = 0 \) is substantially used. This condition is necessarily fulfilled, if the \( N \)-network \( (\tilde{N}, N, \mu) \) is terminal-class conform connected with an arbitrary \( N \)-networks of \( \mathfrak{S}\mathcal{D}_N(\tilde{N}) \).

Figure 26. Determination of the potential representation of the terminal behavior of an \( M \)-network \( N \in \mathfrak{S}\mathcal{D}_M \) by means of an external norator network, where \( M = (M^l)_{l \in L} \) is defined by \( L := \{1, 2, 3\}, M^1 := \{A, B\}, M^2 := \{C, D, E\}, M^3 := \{F, G\} \).
Yet, it is also possible that the \( N \)-terminal behavior of the direct \( M \)-interconnection of an \( M \)-network \((N, M, \mu)\) with a suitable \( M \)-network \((\tilde{N}, M, \tilde{\mu})\) of \( \Theta_M(N, M, \mu) \) which is in some kind “matched” to \((N, M, \mu)\) can be determined, because of some compensation effects, using only the \( N \)-terminal behavior of \((N, N, \mu)\) and \((\tilde{N}, N, \tilde{\mu})\).

The criteria of O. Brune [98] (p. 322) and [14] deliver typical examples of this kind. These criteria are sufficient conditions under which the two-port behavior of some interconnections of linear four-poles can be described by their two-port parameters.

![Figure 27. Parallel connection of \((N, K)\) and \((\tilde{N}, K)\)](image)

**Example III-C.13** Let \( K \) and \( \tilde{K} \) be terminal class families including only one terminal class \( K^{ts} := \{A, B, C, D\} \) and \( \tilde{K}^{ts} := \{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\} \). Furthermore let \((N, K)\) and \((\tilde{N}, K)\) be four-poles with these terminal class families. Their underlying networks are assumed to be skeleton disjoint. Now we will examine their parallel connection. This parallel connection may be defined by an equivalence relation and a choice function such that the equivalence classes \( \{A, \tilde{A}\} \), \( \ldots \), \( \{D, \tilde{D}\} \) are mapped to the nodes \( A \), \( \ldots \), \( D \), resp. while the nodes of their underlying networks \( N \) and \( \tilde{N} \) which are not terminals are left invariant (cf. Fig. 27). Without restriction of generality we identify these networks with the canonical representatives of their terminal behavior. Fig. 28 shows their voltage graphs. The interconnection of \( N \) and \( \tilde{N} \) defined in that manner is denoted as \( \tilde{N} \).

Let now \( H := (H^j)_{j=1,2} \) and \( \tilde{H} := (\tilde{H}^j)_{j=1,2} \) be terminal class families defined by \( H^1 := \{A, B\} \), \( H^2 := \{C, D\} \) and \( \tilde{H}^1 := \{A, \tilde{B}\} \), \( \tilde{H}^2 := \{C, D\} \), resp. For the determination of the terminal behavior of the two-port \((N, H)\) the terminal pairs \( H^1 \) and \( H^2 \) are according to Theorem III-B.22 connected with norators and lastly (cf. Remark II-F.25) with independent voltage sources. Fig. 29 shows the voltage graph of the resulting interconnection. The corresponding current graph differs from the shown voltage graph only with respect of the orientation of the two branches 4 and 5.

The question is now whether it is possible to determine the terminal behavior of \((N, H)\) using only descriptions of the terminal behavior of the two-ports \((N, H)\) and \((\tilde{N}, \tilde{H})\). One of the special cases of that criteria which are ascribed in [98] (p. 322) to O. Brune says that this is indeed possible if the voltages \( u_{test,1} \) and \( u_{test,2} \) delivered by the test configurations shown in Fig. 30 are equal.

![Figure 30. Test configurations for a parallel connection of \((N, K)\) and \((\tilde{N}, K)\)](image)
are represented by the following constitutive equations

\[
\begin{pmatrix}
i_1 \\
i_2 \\
i_3
\end{pmatrix} = \begin{pmatrix}
H_{11} & H_{12} & H_{13} \\
H_{21} & H_{22} & H_{23} \\
H_{31} & H_{32} & H_{33}
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix},
\]

(3)

\[
\begin{pmatrix}
i_1 \\
i_2 \\
i_3
\end{pmatrix} = \begin{pmatrix}
\tilde{H}_{11} & \tilde{H}_{12} & \tilde{H}_{13} \\
\tilde{H}_{21} & \tilde{H}_{22} & \tilde{H}_{23} \\
\tilde{H}_{31} & \tilde{H}_{32} & \tilde{H}_{33}
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix},
\]

(4)

in hybrid form. Fig. 31 and 32 show the voltage graphs of the networks corresponding to these test configurations.

According to Fig. 31 the equations \( u_2 = u_3 = 0, i_3 = i_3 = i_3 = i_3 = 0, u_3 = u_4 = u_4 = u_3 \) and \( u_{\text{test},1} = u_5 - u_3 \) immediately from Kirchhoff’s rules. They imply together with (1) and (2) the relationship \( u_{\text{test},1} = (\tilde{H}_{31} - H_{31})u_0 = 0 \), i.e., \( \forall u_0 u_{\text{test},1} = 0 \Leftrightarrow \tilde{H}_{31} = H_{31} \).

Analogously it follows \( \forall u_0 u_{\text{test},2} = 0 \Leftrightarrow \tilde{H}_{32} = H_{32} \).

To determine the terminal behavior of the two-port \((N', H)\) we will use a system of tree-voltages and loop-currents. For this purpose we choose in the voltage graph a tree with branch set \( \{1, 2, 3, 4\} \) and in the current graph a tree with branch set \( \{1, 2, 3, 4\} \). By means of the corresponding systems of fundamental loop and cut set equations \( u_1 = u_1 = u_4, u_2 = u_2 = u_5, u_3 = u_3 \) and \( i_1 = i_1 - i_1, i_2 = i_2 - i_2, i_3 = i_3 \) we eliminate in (1) and (2) the quantities \( u_1, u_2, u_3, i_1, i_2 \) and \( i_3 \). (Note, the reference orientations for the branch currents \( i_4 \) and \( i_5 \) are oppositely oriented to that one of \( u_4 \) and \( u_5 \).) Two almost trivial elimination steps (the pivot elements are equal to \( +1 \) and the element to be eliminated equal to \(-1\)) and some appropriate reorderings of rows and columns yield the matrix equation

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \tilde{H}_{13} - H_{13} \\
1 & 0 & 0 & -H_{13} & 1 & -H_{31} \\
1 & 0 & -H_{13} & 1 & -H_{31} \\
& & & \tilde{H}_{32} + H_{32} & & \end{pmatrix} \begin{pmatrix}
i_4 \\
i_5 \\
i_4 \\
i_5
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

(3)

Obviously, the last row of (3) implies \( i_3 = 0 \) then and only then, if additionally to the conditions \( H_{31} = \tilde{H}_{31} \) and \( H_{32} = \tilde{H}_{32} \) resulting from Brune’s test the condition \( H_{33} + \tilde{H}_{33} \neq 0 \) is fulfilled. Yet, if this additional condition is fulfilled, then it follows immediately

\[
\begin{pmatrix}
i_4 \\
i_5
\end{pmatrix} = \begin{pmatrix}
\tilde{H}_{11} + \tilde{H}_{11} & \tilde{H}_{12} + \tilde{H}_{12} \\
\tilde{H}_{21} + \tilde{H}_{22} & \tilde{H}_{22} + \tilde{H}_{22} \\
\tilde{H}_{31} + \tilde{H}_{31} & \tilde{H}_{32} + \tilde{H}_{32}
\end{pmatrix} \begin{pmatrix}
u_4 \\
u_5
\end{pmatrix}.
\]

(4)

This result shows that it is under the given assumptions indeed possible to determine the terminal behavior of the two-port \((N', H)\) using exclusively equations describing the terminal behaviors of the two-ports \((N, H)\) and \((N', H')\).

But, if \( H_{33} + \tilde{H}_{33} = 0 \), then \( (H_{33} + \tilde{H}_{33})i_3 = 0 \), i.e. \( 0i_3 = 0 \), implies by no means \( i_3 = 0 \). Therefore an arbitrary loop current can circulate through the branches 3 and 3 and Equation (4) can deduced from the first two rows of (3) only if \( H_{13} = \tilde{H}_{13} \) and \( H_{23} = \tilde{H}_{23} \). The standard example for this situation is the case, where all elements in the third row and third column of the coefficient matrices of the Equations (1) and (2) vanish. The two-ports \((N, H)\) and \((N', H)\) have then a throughout ground and can be reduced to three-poles (cf. Definition III-C.26, more below). □

Obviously, Brune’s criteria are for applications without any noteworthy importance. They are related to nongeneric situations because of unavoidable device tolerances it is practically impossible to realize two-ports \((N, H)\) and \((N', H)\) fulfilling the balance conditions \( H_{31} = \tilde{H}_{31} \) and \( H_{32} = \tilde{H}_{32} \) exactly.

Next we show there are examples where an \( M \)-network \((N', M, \mu) \in \mathcal{M} \) even has a certain special property guaranteeing that this network can be replaced in terminal-class conform interconnections with arbitrary skeleton disjoint \( M \)-networks by an \( M \)-network \((N', M, \mu) \) fulfilling the condition \((N, N, \mu) \in \mathcal{S}_N \) and not only with some “matched” networks.

**Example III-C.14** Let \( M = \{M^j\}_{j \in J} \) be a label class family defined by \( L := \{1\} \) and \( M^j := \{1, 2, 3, 4\} \) and let \( N = (N^j)_{j \in J} \) be a subdivision of \( M \) defined by \( J := \{1, 2\} \) and \( N_1 := \{1, 2\} \) and \( N_2 := \{3, 4\} \).

Fig. 33 shows two \( M \)-equivalent networks of \( E_M \). The network shown in Fig. 33 a) is an ideal two-winding transformer. It is obviously an element of \( \mathcal{S}_N \), too. □
Let $n_G$ be a subdivision of $G$ and $n$ be a label class family as in Example III-B.23 have the same behavior of $\bar{G}$ going back to [100], cf. additionally [47].

Such a special property can be characterized with the help of a notion introduced to the following definition.

**Definition III-C.17** Let $M = (M^l)_{l \in L}$ and $N = (N^j)_{j \in J}$ be label class families and $N$ be a subdivision of $M$.

$\Theta_{M/N}$ denotes the set of all cut sets of the $M$-graph $G_M = (UV, UM, id_{UV})$ which are defined by the partitions $(N^j, U \setminus N^j)$ ($j \in J$) of the node set of $G_M$.

$\mathcal{M}_{M/N}$ denotes that subset of the branch set of $G_M$ defined by $\mathcal{M}_{M/N} := \{ b \in UV \mid \forall (Z+, Z-) \in \Theta_{M/N} b \notin Z+ \land Z^- \}$. □

The transformer equivalences shown in Fig. 33 going back to [100], cf. additionally [47].

**Example III-C.15** Let $M$ and $N$ denote the same label class families as in Example III-C.14.

Fig. 34 b), c), Fig. 34 d) shows three $M$-equivalent OV models of $\mathcal{E}_M$. Their constitutive relations are represented with $V \in \mathbb{R}$ by the following systems of constitutive equations:

\[
\begin{align*}
\dot{i}_1 &= i_2 = 0, & u_3 &= V(u_1 - u_2), \\
\dot{i}_1 &= i_3 = 0, & u_2 &= V u_1, \\
\dot{i}_1 &= 0, & u_2 &= V u_1.
\end{align*}
\]

Obviously, the voltage controlled voltage source shown in Fig. 34 d) is an element of $\mathfrak{G}_N$. □

**Example III-C.16** Let $M = (M^l)_{l \in L}$ be a label class family defined by $L := \{ 1 \}$ and $M^1 := \{ A, B, C, D \}$ and let $N = (N^j)_{j \in J}$ be a subdivision of $M$ defined by $J := \{ 1, 2 \}$, $N_1 := \{ A, B \}$ and $N_2 := \{ C, D \}$. Then the $M$-networks of Example III-B.23 have the same $M$- and the same $N$-terminal behavior. The $M$-network on the left hand side of Fig. 21 is likewise an element of $\mathfrak{G}_N$. □

Figure 33. Equivalent transformer models [100]

Figure 34. Equivalent OV models

The next theorem delivers an equivalent characterization of $M/N$-networks.

**Theorem III-C.20** Let $M = (M^l)_{l \in L}$ and $N = (N^j)_{j \in J}$ be label class families and $N$ be a subdivision of $M$. Furthermore let $\mathcal{N}$ be a network with at least $|UM|$ nodes and let $\mu$ be an injection of $UM = UN$ into the node set of $\mathcal{N}$.

$\mathcal{N}$ and $\mathcal{N}$ is then and only then an $M/N$-network, if for each canonical representative $(\mathcal{N}, N, \mu)$ of the $N$-terminal behavior of $(\mathcal{N}, N, \mu)$ the relationship $(\mathcal{N}, M, \mu) \preceq_M (N, M, \mu)$ holds.
The family of current conveyors of second generation [97] is a class of four poles with a terminal set whose elements are traditionally denoted by $X$, $Y$, $Z$ and $G$ (cf. Fig. 36). Its elements can be considered with a label class family $M := (M^j)_{j \in L}$ defined by $L := \{1\}$, $M^1 := \{X, Y, Z, G\}$ as $M$-networks of $E_M$.

The constitutive relation of that canonical representative shown on the right hand side of Fig. 36 is determined by the constitutive equation

$$ \begin{pmatrix} u_x \\ i_y \\ i_z \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix} \begin{pmatrix} i_x \\ u_y \\ u_z \end{pmatrix} $$

and the list

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$CCII(\alpha, \beta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$CCII^+ := CCII(1, 1)$</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>$CCII^- := CCII(1, -1)$</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>$ICCI^+ := CCII(-1, 1)$</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>$ICCI^- := CCII(-1, -1)$</td>
</tr>
</tbody>
</table>

of the values of the parameters $\alpha$ and $\beta$ defining the members of this family.

Fig. 37 shows two other $M$-equivalent representatives of a $CCII^-$. Let $N := (N^j)_{j \in J}$ be the label class family defined by $J := \{1, 2\}$, $N^1 := \{X, Y, Z\}$ and $N^2 := \{G\}$, then the $M$-network on the right hand side is also an element of $\hat{\mathcal{S}}_N \cap E_N$.

With this notations, the second generation current conveyor $CCII^-$ is an $M/N$-network. $\square$

The next example is related to bondgraph theory. Bondgraphs has been introduced by H. Paynter [73] for interdisciplinary applications of network theory ([44], [113] (for a short introduction cf. [75] and [2])).

Bondgraphs are oriented graphs without selfloops and isolated nodes. They describe the interconnection of multiports. Their nodes represent multiports with, in general, different numbers of ports. Their branches, denoted as bonds, describe the bidirectional exchange of energy between these ports, where each bond connects a pair of ports of different multiports. The orientations of the bonds, traditionally depicted by so-called half-arrows, are the reference directions of this energy exchange. Two variables are assigned to each bond. They represent time-functions whose product deliver the power transported through the corresponding bond. With view of an interpretation of bondgraphs as electrical networks, these time functions are denoted in the following as bond voltages and bond currents, whereas in the bond graph literature at this place the neutral, interpretation-independent notations effort and flow are preferred.

The set of multiports of a bondgraph is partitioned into a class of lossless junction multiports and a complementary class, here denoted as the class of external multiports. The class of junction multiports consists of series and parallel junction

\[ \begin{array}{c}
\begin{array}{c}
X \\
Y \\
G
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
X \\
Y \\
G
\end{array}
\end{array} \]

Figure 36. Circuit symbol and a canonical representative of the terminal behavior of a current conveyor of second generation

\[ \begin{array}{c}
\begin{array}{c}
X \\
Y \\
G
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
X \\
Y \\
G
\end{array}
\end{array} \]

Figure 37. Current conveyor $CCII^-$
multiports. Each port of an external multiport can be only connected with a port of a junction multiport.

If $Z^+$ and $Z^-$ denote the sets of bonds incident at such a junction multiport and oriented toward respectively away from this multiport, then the constitutive relation of a series junction or a parallel junction is defined by the constitutive equations

$$\sum_{b \in Z^+} u_b - \sum_{b \in Z^-} u_b = 0 \quad \forall b,c \in (Z^+ \cup Z^-) \quad i_b = i_c$$

or

$$\sum_{b \in Z^+} i_b - \sum_{b \in Z^-} i_b = 0 \quad \forall b,c \in (Z^+ \cup Z^-) \quad u_b = u_c,$$

resp.

Figure 38. Examples of bondgraph symbols for junction 3-ports, a) series junction 3-port, b) parallel junction 3-port

For the special case $Z^+ := \{a, b\}$ and $Z^- := \{c\}$ these junction 3-ports are represented as shown in Fig. 38. The use of the symbols $\nabla$ or $O$ to denote series respectively parallel junctions goes back to [75]; in bondgraph literature the symbols $I$ and $O$, resp., are used instead of $\nabla$ and $O$ to denote these junctions.

![example of a bondgraph representing a network](image)

Fig. 39 shows a simple example of a bondgraph representing the interconnection of a resistive two-port with an inductive and a capacitive one-port.

**Example III-C.23** To assign to a bondgraph an electrical network the external multiports are interpreted as elementary multiports in the sense of Definition III-A.2 and the junction multiports are usually replaced by some special lossless networks consisting of short circuit branches only. The networks $N_\nabla$ and $N_O$ shown in Fig. 40 are examples of such a realization of a series and a parallel junction 3-port, resp. All other cases can be realized by similar constructions.

Let $N$ be a label class family defined by $N := (N_j)_{j \in J}$, $J := \{1, 2, 3\}$, $N^1 := \{A, B\}$, $N^2 := \{C, D\}$ and $N^3 := \{E, F\}$ and let furthermore $\mu := \text{id}_{N\!N}$. Then it is immediately to see, that the $N$-terminals of $N$ are $\{A, B\}$ and $\{C, D\}$ and $\{E, F\}$ can be represented by canonical representatives $(\hat{N}_\nabla, N, \mu)$ or $(\hat{N}_O, N, \mu)$ with branch set $\hat{\Delta} := \{a, b, c\}$ whose constitutive relation can be described, as a special case of Equation (5) or (6), resp., by

$$u_a + u_b - u_c = 0, \quad i_a = i_b, \quad i_a = i_c$$

or

$$i_a + i_b - i_c = 0, \quad u_a = u_b, \quad u_a = u_c,$$

resp.

![realizations of junction 3-ports](image)

Let $M$ be the label class family defined by $M := (M_\!l)_{l \in L}$, $L := \{1\}$, $M^1 := \{A, B, C, D, E, F\}$, then it is also immediately to see, that the $M$-labeled networks $(\hat{N}_\nabla, M, \mu)$ and $(\hat{N}_O, M, \mu)$ have not the same $M$-terminal behavior. Similarly, the $M$-networks $(\hat{N}_O, M, \mu)$ and $(\hat{N}_O, N, \mu)$ have different $M$-terminal behaviors. For instance, the voltage between the terminals $A$ and $C$ of $(\hat{N}_\nabla, M, \mu)$ is always equal to zero, whereas this voltage in $(\hat{N}_\nabla, M, \mu)$ can have arbitrary values. Moreover, the branch sets of the canonical representatives of the $M$-terminal behavior of $(\hat{N}_\nabla, M, \mu)$ and $(\hat{N}_O, M, \mu)$ include five rather than three branches and a representation of their constitutive relations by means of constitutive equations needs likewise at least five linearly independent scalar equations and not only three. That means, the triples $(\hat{N}_\nabla, M, \mu)$ and $(\hat{N}_O, N, \mu)$ are $M$-networks, but the quadruples $(\hat{N}_\nabla, M, N, \mu)$ and $(\hat{N}_O, M, N, \mu)$ are not $M/N$-networks.

Yet, the canonical representatives $(\hat{N}_\nabla, N, \mu)$ and $(\hat{N}_O, N, \mu)$ of the $N$-terminal behavior of $(\hat{N}_\nabla, N, \mu)$ and $(\hat{N}_O, N, \mu)$ are $M/N$-networks according to Theorem III-C.20.

These observations were the starting point for the reformulation of bondgraph theory presented in [89] on the base of [81], [53]. The central idea behind this reformulation is the fact that the Equations (6) and (5) can be interpreted as the constitutive equations of ideal transformer multiports of first and second kind introduced in [13], [14] by W. Cauer.
In that manner it is possible to assign to each bondgraph a network in the sense of Definition II-B.2. Such a network, denoted in [89] as a Paynter network, is an interconnection of elementary multiports. Each component of the graphs of a Paynter network consists of two parallel branches. These branches form in its voltage or current graph a similarly directed cut set or a similarly directed loop, resp.

Obviously, the graphs of a Paynter network are planar. Therefore each Paynter network is dualizable, and this dual network is likewise a Paynter network. That implies that to each bondgraph exists a dual one. This fact explains why the differences in the analogies between some physical quantities, each bondgraph exists a dual one. This fact explains why the differences in the analogies between some physical quantities, do not cause difficulties or contradictions.

By means of this reformulation of bondgraph theory all seeming contradictions, sometimes discussed in the literature (e.g. [76], [74]), are removed.

Let \( Z \) be the branch set of a Paynter network \( \mathcal{N} \), let \( Z^{\text{ext}} \) be the branch set of its external multiports and \( Z^{\text{conn}} := Z \setminus Z^{\text{ext}} \), then the subnetwork \( \mathcal{N}_{Z^{\text{conn}}} \) is the connection network of this Paynter network. It defines in a natural manner a multiport with \( n := |Z^{\text{ext}}| \) terminal pairs.

As said above, the junction multiports of a bondgraph correspond in a Paynter network to some ideal transformers. According to [13], [14], [5] these transformer multiports can be equivalently replaced by interconnections of ideal two-winding transformers. Fig. 41 shows two examples of this kind, where all their transformers have a 1 : 1 turns ratio. (Note the differences between the reference directions for \( u_c \) and \( i_c \) in comparision to that ones of \( u_a \), \( i_a \) and \( u_b \), \( i_b \) in these figures. For the the so-called dot convention, used here at the graphical representation of the ideal transformers, we refer to [28].)

With the notations introduced in Example III-C.23 it is easy to prove that the pair \((\mathcal{N}_1, M, \mu)\) and \((\mathcal{N}_2, M, \mu)\) of \( M \)-labelled networks as well as that one \((\mathcal{N}_0, M, \mu)\) and \((\mathcal{N}_0, M, \mu)\) are \( M \)-equivalent. The 4-tuples \((\mathcal{N}_1, M, N, \mu)\) and \((\mathcal{N}_0, M, N, \mu)\) deliver therefore examples of \( M/N \)-networks.

If in a network interpretation of a bondgraph the ideal transformers of first and second kind, representing the parallel respectively series junctions, are replaced by interconnections of ideal two-winding transformers, then the result of this replacement is likewise a Paynter network.

Let now \( \mathcal{N} \) be a given Paynter network whose connection network consists of ideal two-winding transformers. Then it is in some cases, but not always (!), possible to eliminate all these two-winding transformers without changing the terminal behavior of the corresponding connection networks. A simple example of this kind is the bondgraph and the associated network shown in Fig. 39 a) and b), resp.

Obviously, this network is a resistive one. Yet, because it includes an ideal transformer, it is not free of coupled branches. Therefore it is not astonishing that this network does not show the no-gain property (cf. [76], [74]).

For the special case defined by the assignments \( i_{1}^{\text{pr}}(t) := 5 \) \((t \in T := \mathbb{R})\), \( G_2 := G_4 := 1 \) and \( G_3 := 2 \), the instantaneous values of the unique solution of \( \mathcal{N} \) are determined by the
Under the assumptions of the last theorem and with the notations used there, a quadruple \((N, M, N, \mu)\) is an \(M/N\)-network if and only if the potential representation \(V_M^\phi\) of the \(M\)-terminal behavior of \((N, M, \mu)\) fulfills the conditions:

\[
\forall (u^\phi, i^\phi) \in S_M^{\phi,i} \left[ (u^\phi, i^\phi) \in V_M^{\phi,i} \land \forall i \in L \forall m,n \in M \right.
\]
\[
v_m^\phi = v_n^\phi \implies (u^\phi + v^\phi, i^\phi) \in V_M^{\phi,i}
\]

\[
\forall (u^\phi, i^\phi) \in V_M^{\phi,i} \left[ \sum_{n \in M} v_n^\phi = 0 \right.
\]

The terminal classes \(\mu^N(j) \ (j \in J)\) correspond to the components of multipoles introduced axiomatically in [86].

Using the label functions of the \(M/N\)-networks it is possible to transfer the notion of an \(M/N\)-network to the class of multipoles in the sense of Definition III-A.1.

**Definition III-C.25** Let \(M = (M^I)_{i \in I}\) and \(N = (N^J)_{j \in J}\) be label class families and \(N\) a subdivision of \(M\). Let \((N, M, \mu)\) be an \(M\)-network and let \(K\) and \(H\) be the terminal class families defined by \(K := (\mu(M^I))_{i \in I}\) and \(H := (\mu(N^J))_{j \in J}\), resp.

The multipole \((N, H)\) is denoted as intrinsic with respect to \(K\), if \((N, M, N, \mu)\) is an \(M/N\)-network.

This definition delivers an explication for a often only intuitively used term, cf. e.g., [20].

**Definition III-C.26** Let \(N\) be an \(M\)-network.

The terminal set of \(N\) is reducible, if the set \(\mathfrak{R}_M(N)\) of the canonical representatives of its \(M\)-terminal behavior includes an \(M\)-network \(N\) which includes at least one short circuit branch.

**Theorem III-C.27** If the terminal set of a labelled network is reducible, then it is possible to reduce the number of its terminals by a contraction of these short circuit branches.

If the terminal set produced in that manner includes now a terminal which is at most connected with some other terminals by open circuit branches, then this isolated terminal can be additionally deleted.

If the terminal set of a labelled network is reducible, then the number of its terminals can be reduced by contracting appropriate short-circuit branches. If one of the terminals generated in such a way is connected with the remaining terminals only by open circuit branches, then these isolated terminals could also be omitted completely as already discussed further above.

In this connection the question results whether to each labelled network exists a canonical representative including maximally many open and short circuit branches.

### IV. Concluding Remarks

Based on the formalization of network theory outlined in Section II we have introduced multipoles and multiports and developed the fundamentals of a general theory of terminal behavior of networks. In relation to our former work to these topics the representation has been substantially standardized and simplified.

We have shown that network-theoretical models for arbitrary technical devices can be developed, if the behavior of these devices can be described at least at their terminals in terms of voltages and currents.

The theory of transmission lines delivers typical examples at which network-theoretical models are able to represent only the terminal behavior of such a device.

An important byproduct of our investigations (cf. also [86]) is the insight that degenerated resistive networks, such as nullators, norators and fixators [12], [11], [10] or the voltage and current mirrors regarded in the newer literature [3], are by no means useless or even “pathological” idealizations. They deliver useful tools as well for the development of network theory as for the development of algorithms for network analysis.

Moreover, the introduction of the spaces of branch voltage and branch current values as one-dimensional normed oriented real linear spaces leads in a natural manner to the definition of multi-dimensional networks necessary for some interdisciplinary applications of network theory. Furthermore, we owe to the theory of terminal behavior the deciding hints for a reformulation of bondgraph theory removing all seeming contradictions sometimes discussed in the literature.

### APPENDIX A

**Set-Theoretical Notations**

Binary relations are considered as sets of ordered pairs. If \(R\) is a binary relation, then \(R^{-1} := \{(y, x) \mid (x, y) \in R\}\) is its inverse relation. Functions and maps are consequently considered as special binary relations realizing a unique assignment from its domain into its range. As usual in set theory, we denote for any two sets \(X\) and \(Y\) the set of all mappings from \(X\) to \(Y\) by \(Y^X\). A mapping \(f \in Y^X\) is also denoted by \(f : X \to Y\). The set dom \(f := X\) is the domain of \(f\) and the set \(\text{rng} f := \{y \in Y \mid \exists x \in X \ y = f(x)\}\) is the range of \(f\). For each \(f \in Y^X\) and \(X' \subseteq X\) we denote with \(f|X' := f \cap X' \times Y\) the restriction of \(f\) to \(X'\). The notion of a family is used as a synonym for the notion of a function. If \((X_i)_{i \in I}\) is a family, then the index set \(I\) is the domain of this function and \(X_i\) denotes the function value of this function at \(i \in I\). For each \(I' \subseteq I\) the restriction of \((X_i)_{i \in I}\) to \(I'\) is denoted as \((X_i)_{i \in I'}\).

The cardinal number of any set \(X\) is denoted by \(|X|\). If \(X\) is a finite set, then \(|X|\) is equal to the number of its elements. (For additional details cf. [39], [30], [96].)
An oriented graph is a triple \((G, K, A)\) of two sets and a map, satisfying the following conditions: (i) \(G\) and \(K\) are disjoint finite sets, (ii) \(K = \emptyset \Rightarrow G = \emptyset\), and (iii) \(A : G \rightarrow K \times K := \{(v, w) \mid v, w \in K\}\). The elements of \(G\) are the branches, \(K\) itself is the branch set. The elements of \(K\) are the nodes of this graph, \(K\) itself is called its node set. The incidence map \(A : G \rightarrow K \times K\) assigns to every branch an ordered pair of nodes. A branch \(b \in G\) is incident with a node \(n \in K\) if \(\exists_{n \in K} A(b) = (n, m) \Rightarrow A(b) = (n, m)\). For \(b \in G\) and \(m, n \in K\) it is said that branch \(b\) is oriented from \(m\) to \(n\) if \(\exists_{A(b) = (m, n)}\). The sets \(A^+ := \{(v, w) \mid \exists_{w \in K} A(b) = (v, w)\}\) and \(A^- := \{(b, w) \mid \exists_{v \in K} A(b) = (v, w)\}\) are the incidence relations of the oriented graph \((G, K, A)\). Let \(G = (G, K, A)\) and \(\bar{G} = (\bar{G}, \bar{K}, \bar{A})\) be oriented graphs. The ordered pair \((\zeta, \kappa)\) is an isomorphism between \(G\) and \(\bar{G}\) if \(\zeta\) and \(\kappa\) are bijections \(\zeta : G \rightarrow \bar{G}\), \(\kappa : K \rightarrow \bar{K}\) with \(A \circ \zeta = (\kappa \circ \kappa) \circ \Lambda\) where \(\kappa \times \kappa\) is defined for all \((v, w) \in K \times K\) by \((\kappa(v), \kappa(w))\).

We denote the oriented loops and the oriented cutsets as ordered pairs \((G, \bar{G})\) of disjoint subsets of the branch set of an oriented graph. Let \((G, \bar{G})\) be an oriented loop or cut set. A branch \(b \in G \cup \bar{G}\) is similarly directed (oppositely directed) with respect to the orientation of the oriented loop or cut set, resp., if \(b \in G\) \((b \in \bar{G})\).

Let \(G\) be an oriented graph with node set \(K\) and \(v \in K\). The cut defined the pair \((\{v\}, K \setminus \{v\})\) of corresponding node sets is called the incidence cut of \(v\).

If \((\zeta, \kappa)\) is an isomorphism between \(G\) and \(\bar{G}\), then the assignment \((G, \bar{G}) \mapsto (\zeta(G), \zeta(G))\) defines bijections as well between the sets oriented loops of \(G\) and \(\bar{G}\) as between the sets of cutsets of these graphs.

Let \(G = (G, K, A)\) and \(\bar{G} = (\bar{G}, \bar{K}, \bar{A})\) be oriented graphs. We denote a bijection \(\zeta : G \rightarrow \bar{G}\) as an Whitney-isomorphism between \(G\) and \(\bar{G}\) and \(\zeta\) is called Whitney-isomorphic to \(\bar{G}\) if the assignment \((G, \bar{G}) \mapsto (\zeta(G), \zeta(G))\) defines bijections as well between the sets oriented loops of \(G\) and \(\bar{G}\) as between the sets of cutsets of these graphs.

A nonoriented graph is a triple \((G, K, A)\) satisfying the conditions (i), (ii), as above, and the condition (iii') \(A : G \rightarrow K \& K := \{v, w \mid v, w \in K\}\). The incidence map assigns here to every branch an unordered node pair, i.e. a set of one or two nodes. A branch \(b \in G\) is incident with a node \(n \in K\) if \(\exists_{m \in K} A(b) = \{m, n\}\), where the case \(m = n\) is admitted.

A hypergraph is a triple \((G, K, A)\) satisfying the conditions (i), (ii), as above, and the condition (iii'') \(A : G \rightarrow \mathfrak{P}(K) \setminus \{\emptyset\}\). The incidence map assigns here to every branch a nonvoid subset of nodes. A branch \(b \in G\) is incident with a node \(n \in K\) if \(n \in A(b)\).

Let \(G = (G, K, A)\) be an oriented graph and \(\bar{G} = (\bar{G}, \bar{K}, \bar{A})\) be a nonoriented graph. \(\bar{G}\) is the underlying nonoriented graph of \(G\) if \(Z = \bar{Z}\), \(K = \bar{K}\), and every branch \(b \in G\) is incident with the same nodes in \(\bar{G}\) and \(\bar{G}\). Two oriented graphs with the same branch and node set differ at most with respect to their orientation if they have the same underlying nonoriented graph.