Numerical Verification of Five Solutions in Two-transistor Circuits

Yusuke Nakaya†, Shin’ichi Oishi†, Tetsuo Nishi† and Martin Claus‡

†Faculty of Science and Engineering, Waseda University
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
‡Department of Electrical Engineering and Information Technology, Dresden University of Technology
D-01062 Dresden, Germany

Email: nakaya@waseda.jp, oishi@waseda.jp, nishi-t@waseda.jp, claus@iee.et.tu-dresden.de

Abstract—This paper is concerned with the number of operating points in two-transistor circuits. Recently, a new method was proposed for analyzing two-transistor circuit and an Ebers-Moll BJT circuit possessing five operating points was shown numerically. In this paper, applying the technique of numerical validation, we verify the numerical results rigorously and prove that the circuit have more than three operating points rigorously.

1. Introduction

One of the most fundamental problems on resistive transistor circuits is to find the maximum number of solutions of \( m \)-transistor circuits and it has long been studied [1]–[6] since the famous Nielsen-Willson theorem on a unique solution [1]–[9]. There are some conjectures on the maximum number of solutions. For example, it may be \( 2^m - 1 \), etc. It was once proposed as “a Challenging Problem” in the field of Nonlinear Circuits and Systems, but it still remains unsolved even for the simplest case of \( m = 2 \).

The number of solutions of course depends on various factors, for example, types of transistors (BJT or MOST), modeling of a transistor (Ebers-Moll or Gummel-Poon), performance of circuit simulators, the characteristics of a diode function \( f \) (exponential function, piecewise-linear function, etc) in transistor models. The difficulties of the problem can be seen from [10]–[12].

One of the most well-known results so far, which has long been believed to be true, was that the maximum number of solutions of two-transistor circuits is three. The proof by Lee [2]–[3] is extremely complex to fully understand. Afterwards Goldgeisser et al [7] showed that Lee’s theorem is not logically true for MOST circuits and then constructed a five-solution MOST circuit (a modified latch circuit having the feedback structure). After that Shou et al [8] constructed even for BJT circuits a five-solution circuit by using circuit simulator model. Recently Claus [9] proposed a new analysis method for a two-transistor circuit which is based on the notion of terminal behavior of a network. Applying this method on a MOST circuit and an Ebers-Moll BJT circuit he showed numerically that these circuits possess a least five solutions. Furthermore, the results are verified with homotopy methods [9] realizable with standard circuit simulators.

As stated above, all five-solution circuits so far found are derived based on theoretically predicted observations and are numerically examined. However in general the numerical computation always involves some numerical errors and the accuracy of computation depends on the ability of circuit simulators, etc. Indeed the numerical results by Spice and Matlab differs considerably (private communication). So if we intend to confirm these results surely, the numerical results have to be verified carefully.

Though the numerical validation is usually considered to require much computation time, we are recently developing various effective tools for fast numerical validation. So the aim of this paper is to give a methodology for validating the numerical results rigorously and to show that, as an example, the Claus’ results can be rigorously proved to be true by numerical analysis.

2. Fundamental Technique for the Numerical Verification

Let us consider about numerical verification of a solution for a system of nonlinear equations

\[ f(x) = 0, \]  

where \( f : \mathbb{R}^n \to \mathbb{R}^n \). For the purpose, we can apply Krawczyk’s method which is known as one of the efficient method to prove the existence of the solution [14]. In this section, this Krawczyk’s method is introduced for the verification of solutions in the following section.

For an interval \( T \) whose center is \( c \), an interval ma-
matrix $M$ and an interval mapping $K$ are defined as

$$M = E - L^{-1}F'(T)$$

(2)

and

$$K(T) = c - L^{-1}f(c) + M(T - c)$$

(3)

respectively, where $E$ is an $n \times n$ unit matrix, $L$ is an approximate matrix of $f'(c)$, and $F^*$ is an interval inclusion of $f'$. Then, if the condition

$$K(T) \subset T$$

(4)

holds, there exists a unique solution $x^*$ of the nonlinear equations (1) in the interval $T$.

Here, let $c$ be an approximate solution of (1) which is obtained by an appropriate method. Define $T$ as

$$T = c + X$$

(5)

where, $X$ is described as

$$X = [-c, e],$$

(6)

and $e$ is defined as

$$e = 2\|L^{-1}f(c)\|_{\infty}.$$  

(7)

Using the Krawczyk operator (3), define the operator $H$ as

$$H = K(T) - c$$

$$= K(c + X) - c$$

$$= -L^{-1}f(c) + (E - L^{-1}F'(c + X)) X.$$  

(8)

At this time, the condition (4) is equivalent to

$$H \subset X.$$  

(9)

Therefore, in the following sections, we apply the operator $H$ and check the condition (9) for proving the existence of solution.

### 3. Analysis of Two-transistor Circuit

In [9], Claus has analyzed a two-transistor circuit shown in figure 1, and it is reported that this circuit has five operating points under certain conditions. In this section, we analyze this two-transistor circuit, and attempt to prove that this circuit has certainly five operating points by calculating with error verification.

For calculating operating points of the circuit numerically, we apply the Ebers-Moll model as a model of transistor. A transistor is represented as figure 2 by Ebers-Moll model, and terminal current is given as

$$\begin{bmatrix}
-I_1 \\
-I_2 
\end{bmatrix} = \begin{bmatrix}
1 & -\alpha_f & 1 \\
-\alpha_r & 1 & 0 
\end{bmatrix} \begin{bmatrix}
f_1(V_1) \\
f_2(V_2)
\end{bmatrix},$$

(10)

where

$$f(V) = \begin{bmatrix}
\frac{V}{V_T} & \frac{V}{V_T} & -1 \\
\frac{V}{V_T} & \frac{V}{V_T} & -1 \\
\frac{V}{V_T} & \frac{V}{V_T} & -1 \\
\frac{V}{V_T} & \frac{V}{V_T} & -1 
\end{bmatrix},$$

(11)

Applying (10), the equation of the circuit shown in figure 1 is given as a system of nonlinear equations as follows:

$$Tf(V) + GV + J = 0,$$

(12)

where

$$T = \begin{bmatrix}
1 & -\alpha_r & 0 & 0 \\
-\alpha_f & 1 & 0 & 0 \\
0 & 0 & 1 & -\alpha_r \\
0 & 0 & -\alpha_f & 1 
\end{bmatrix},$$

(13)
G = \begin{bmatrix} 2G_b+G_c - (G_b+G_c) & -2G_b & G_b \\ -2G_b & G_b & 2G_b+G_c - (G_b+G_c) \\ G_b & 0 & G_b+G_c \end{bmatrix} \quad (15)

V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}, \quad (16)

J = \begin{bmatrix} G_cV_{cc} \\ G_bV_s - G_cV_{cc} \\ G_bV_s - G_cV_{cc} \end{bmatrix}, \quad (17)

and \( G_b, G_c \) represent \( \frac{1}{R_b}, \frac{1}{R_c} \) respectively.

We analyze the case that the values of \( R_b, R_c \) and \( V_{cc} \) are 10\,kΩ, 5\,kΩ and \(-5\,\text{V}\) respectively and \( V_s \) works as a parameter. On the other hand, for the parameters of transistor, we choose \( \alpha_f = 0.99, \alpha_r = 0.5, \) and we consider following two cases: (a) \( I_s = 10^{-9}(\text{A}), V_T = 0.053(\text{V}) \), (b) \( I_s = 10^{-6}(\text{A}), V_T = 0.102(\text{V}) \).

On the first step, we calculate the solutions of the equation (12) by normal numerical calculation, namely without verifying errors. By solving the equation (12) for different values of \( V_s \), we obtain following bifurcation diagrams. Figure 3 shows the case (a), and Figure 4 shows the case (b). On these bifurcation diagrams, the number of operating points is indicated as the number of intersection points of the bifurcation diagram with a vertical line corresponding to a specific value of \( V_s \). It is obviously recognized that there exists an interval for \( V_s \) in which there are five operating points for both cases.

From these results, on the next step, we prove rigorously that the two-transistor circuit shown in figure 1 has five operating points. From figure 3 and 4, we consider the case \( V_s = -0.64(\text{V}) \) for case (a) and \( V_s = -0.44(\text{V}) \) for case (b). For the purpose, we apply Krawczyk’s method to the nonlinear equation (12). As approximate solutions for equation (12), we choose intersection points of the bifurcation diagram and vertical line of \( V_s \). Applying Krawczyk’s method, five solutions are calculated rigorously for each cases of (a) and (b). These results are shown in table 1 and 2. From this results, it is rigorously proved that the two-transistor circuit shown in figure 1 have at least five operating points.

4. Conclusion

In this paper, we were concerned with the number of operating points in two-transistor circuits. For proving that there are some two-transistor circuits which have more than three operating points, an Ebers-Moll BJT circuit was analyzed. Applying the technique of
Table 1: Verified solutions of case (a) on $V_s = -0.64(V)$

<table>
<thead>
<tr>
<th>No.</th>
<th>$V_1$</th>
<th>$V_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.74231646, 0.74231648</td>
<td>0.61988727, 0.61988729</td>
</tr>
<tr>
<td>#2</td>
<td>0.73551253, 0.73551255</td>
<td>0.57555292, 0.57555294</td>
</tr>
<tr>
<td>#3</td>
<td>0.72929862, 0.72929864</td>
<td>0.47145515, 0.47145517</td>
</tr>
<tr>
<td>#4</td>
<td>0.72180713, 0.72180715</td>
<td>0.00333858, -0.00333856</td>
</tr>
<tr>
<td>#5</td>
<td>0.70358962, 0.70358964</td>
<td>-0.72180713, -0.72180715</td>
</tr>
</tbody>
</table>

Table 2: Verified solutions of case (b) on $V_s = -0.44(V)$

<table>
<thead>
<tr>
<th>No.</th>
<th>$V_1$</th>
<th>$V_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.72888014, 0.72888016</td>
<td>0.51630182, 0.51630184</td>
</tr>
<tr>
<td>#2</td>
<td>0.70816937, 0.70816939</td>
<td>0.40346507, 0.40346509</td>
</tr>
<tr>
<td>#3</td>
<td>0.69521428, 0.69521430</td>
<td>0.24620354, 0.24620356</td>
</tr>
<tr>
<td>#4</td>
<td>0.6766788, 0.6766790</td>
<td>-0.20824493, -0.20824495</td>
</tr>
<tr>
<td>#5</td>
<td>0.61902390, 0.61902392</td>
<td>-1.36664114, -1.36664112</td>
</tr>
</tbody>
</table>

numerical validation, we verified numerically that the circuit have at least five operating points rigorously.

References


